LDU decomposition

We can instead write $A = LDU$ where L is lower triangular, D is diagonal, and U is upper triangular. Here, L and U have 1's on the diagonal.

Question What does a diagonal matrix D do to a matrix A?

$$DA = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} d_1 a_{11} & \cdots & d_1 a_{1n} \\ d_2 a_{21} & & \vdots \\ \vdots & & \\ d_n a_{n1} & \cdots & d_n a_{nn} \end{bmatrix}$$

We see that the resulting matrix is the same as A but each row i is multiplied by d_i.

To get the LDU-decomposition, we have

$$LU = L \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{nn} \end{bmatrix}$$

$$= L \begin{bmatrix} u_{11} & 0 & \cdots & 0 \\ 0 & u_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & u_{nn} \end{bmatrix} \begin{bmatrix} 1 & \frac{u_{12}}{u_{11}} & \cdots & \frac{u_{1n}}{u_{11}} \\ 0 & 1 & \frac{u_{23}}{u_{22}} & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

Using the LU decomposition to solve linear systems

Consider $A\vec{x} = \vec{b}$, and $A = LU$, so

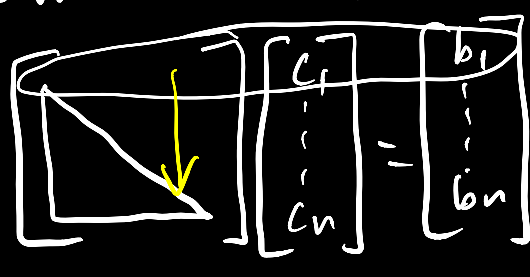
$$LU\vec{x} = \vec{b}.$$

Let $\vec{c} = U\vec{x}$. Then we have the system

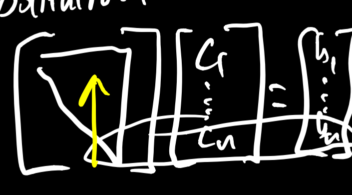
$$L\vec{c} = \vec{b} \quad , \quad \vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

which we can solve using forward substitution. Then use back substitution to solve $U\vec{x} = \vec{c}$.

Forward Substitution



Backward Substitution



Ex
$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

We found in a previous example that that

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 1/3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & -1/3 \end{bmatrix}$$

Let

$$\vec{c} = U\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Now
$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 1/3 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Forward substitution:

$$c_1 = 1$$
$$c_2 = 2 - \frac{1}{2} = \frac{3}{2}$$
$$c_3 = 3 - \frac{1}{2} - \frac{1}{2} = 2$$

So $\vec{c} = \begin{bmatrix} 1 \\ 3/2 \\ 2 \end{bmatrix}$

Now,

$$U \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{c}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & -1/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \\ 2 \end{bmatrix}$$

Back substitution:

$$z = -6$$
$$y = \frac{2}{3}\left(\frac{3}{2} + 6\right) = 5$$

$$x = \frac{1}{2}(1-5) = -2$$

so $\vec{x} = \begin{bmatrix} -2 \\ 5 \\ -6 \end{bmatrix}$.

2.7 Transposes and Permutations

<u>Definition</u> The transpose of A, denoted A^T is the matrix whose entries are

$$(A^T)_{ij} = A_{ji}$$

<u>Remark</u> 1. The columns of A^T are the rows of A.

2. If A is $m \times n$, then A^T is $n \times m$.

Ex $A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 2 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{bmatrix}$$

Transpose rules:

Sum $(A+B)^T = A^T + B^T$

Product $(AB)^T = B^T A^T$

Inverse $(A^{-1})^T = (A^T)^{-1}$

Proof of Product Transpose rule:

Let $C = AB$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Then $(C^T)_{ij} = c_{ji}$

$$= \sum_{k=1}^{n} a_{jk} b_{ki}$$

$$= \sum_{k=1}^{n} b_{ki} a_{jk}$$

Let $D = A^T$ and $E = B^T$

$d_{ij} = a_{ji}$ $\quad$ $e_{ij} = b_{ji}$

$$(C^T)_{ij} = \sum_{k=1}^{n} e_{ik} d_{kj}$$

$$C^T = ED$$

$$= B^T A^T$$

$$(AB)^T = B^T A^T$$

The Meaning of Inner Products

$\vec{x}, \vec{y} \in \mathbb{R}^n$.

$\vec{x}^T\vec{y}$ is the dot product or inner product

$\vec{x}\vec{y}^T$ is the rank one product or outer product

Recall
- $\vec{x}^T\vec{y}$ is a number
- $\vec{x}\vec{y}^T$ is a matrix

A more mathematical definition of the transpose:

A^T is the matrix that makes these two inner products equal

$$(Ax)^T y = x^T (A^T y)$$

That is,

Inner product of $A\vec{x}$ with $\vec{y}$

= Inner product of $\vec{x}$ with $A^T\vec{y}$

__Ex__

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

We want find A^T so that

$$(A\vec{x})^T\vec{y} = \vec{x}^T(A^T\vec{y})$$

$$A\vec{x} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + 2x_2 + x_3 \\ x_1 - x_3 \end{bmatrix}$$

dot product

$\underbrace{\vec{x}^T}_{\text{row vector}}\vec{y}$

$(A\vec{x})^T\vec{y}$ is $(A\vec{x})\cdot\vec{y}$

$$(A\vec{x})^T\vec{y} = \underline{y_1}(x_1 + \underline{2}x_2 + x_3) + \underline{y_2}(x_1 - x_3)$$

$$= x_1(y_1 + y_2) + x_2(2y_1) + x_3(y_1 - y_2)$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} y_1 + y_2 \\ 2y_1 \\ y_1 - y_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \vec{x}^T(A^T\vec{y})$$

$$A^T = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{bmatrix}$$

Symmetric Matrices

Definition A matrix is symmetric if $S^T = S$.
(i.e. $s_{ij} = s_{ji}$).

Ex

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Remarks 1. The inverse of a symmetric matrix is symmetric.

2. The products A^TA and AA^T are symmetric since

$$(A^TA)^T = A^T(A^T)^T = A^TA$$

$$(A^TA)^T = A^TA$$

If S is a symmetric matrix, then if $S = LDU$ requires no row exchanges, $U = L^T$. We get the symmetric factorization

$$S = LDL^T$$

Permutation Matrices

Recall A permutation matrix P has the rows of the identity I in any order.

Ex $P_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$P_{21} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

The inverse of P is P^T. That is,

$$PP^T = I.$$

The $PA = LU$ Factorization

When reducing A to an upper triangular system, we used elimination matrices. If a zero pivot appeared, a permutation was required.

Where should we collect the permutations P_{ij}?

1. Row exchanges can be done in advance. Then

$$PA = LU.$$

2. Row exchanges after elimination

Then, $$A = L_1 P_1 U_1$$

Block Matrices and Block Multiplication

Block matrix:

$$A = \left[\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \end{array}\right]$$

Ex

$$A = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] = \begin{bmatrix} I & I \\ I & I \end{bmatrix}$$

Block Multiplication

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \end{bmatrix}$$