Chapter 3: Vector Spaces and Subspace

3.1 Spaces of Vectors

So far, we have considered vectors in $\mathbb{R}^n$, and we imagined them to be arrows with distinct length and direction. We can now take this idea to a more "fundamental" or abstract level.

Definition A vector space consists of the following

1. A field F of scalars
2. A set V of objects, called vectors

3. An operation called vector addition so that for $\vec{v}, \vec{w} \in V$ $\vec{v} + \vec{w} \in V$ and

(a) addition is commutative
(b) addition is associative
(c) There is a unique vector $\vec{0} \in V$ called the zero vector so that $\vec{0} + \vec{v} = \vec{v}$ for all $\vec{v} \in V$

(d) For each $\vec{v} \in V$, there is a unique vector $-\vec{v} \in V$ so that $\vec{v} + (-\vec{v}) = \vec{0}$

4. An operation called scalar multiplication which associates

each scalar $c \in F$ and vector $\vec{v} \in V$ a vector $c\vec{v} \in V$ so that

(a) $1 \cdot \vec{v} = \vec{v}$ for all $\vec{v} \in V$

(b) $(c_1 c_2)\vec{v} = c_1 (c_2 \vec{v})$

(c) $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$

(d) $(c_1 + c_2)\vec{v} = c_1\vec{v} + c_2\vec{v}$

Ex 1 $\mathbb{R}^n$

Let $\vec{x} \in \mathbb{R}^n$. Then $\vec{x}$ is an n-tuple with $\vec{x} = (x_1, x_2, \ldots, x_n)$ and each $x_i \in \mathbb{R}$. The dimension is n.

Ex 2 $\mathbb{C}^n$

Let $\vec{x} \in \mathbb{C}^n$. Then $\vec{x}$ is an n-tuple with $\vec{x} = (x_1, x_2, \ldots, x_n)$

and each $x_i \in \mathbb{C}$. The dimension is n.

Some less familiar examples:

Ex 3 $\mathbb{R}^{m \times n}$

Let $M \in \mathbb{R}^{m \times n}$. Then M is an $m \times n$ matrix with entries $m_{ij} \in \mathbb{R}$. We have seen that $\mathbb{R}^{m \times n}$ satisfies the requirements to be a vector space. The dimension is mn.

Ex 4 The space of real-valued functions $f(x)$, call it X.

Let $f \in X$. We can check the conditions required for X to be a vector space. Here, the space is infinite-dimensional.

Subspaces

Within a vector space, there are smaller spaces which are vector spaces on their own.

Definition A subspace W of a vector space V is a set of vectors (including $\vec{0}$) so that if $\vec{v}, \vec{w} \in W$ and c is a scalar

(i) $\vec{v} + \vec{w} \in W$

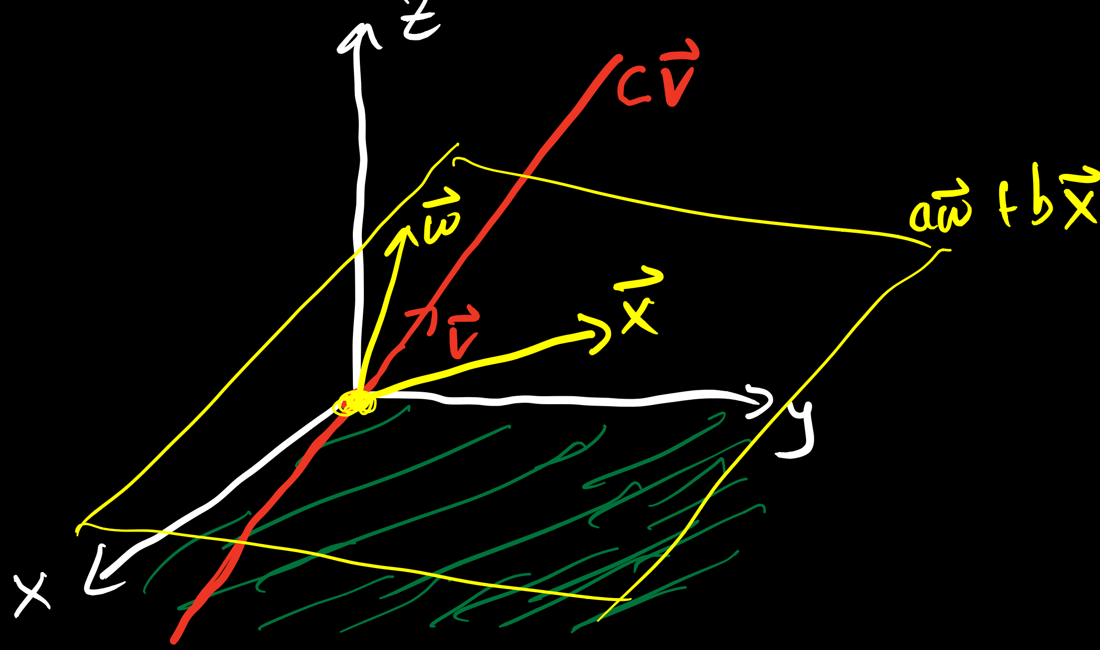
(ii) $c\vec{v} \in W$

Remark 1. All linear combinations stay in the subspace

2. Every subspace contains the zero vector.

<u>Ex 1</u> Consider the vector space $\mathbb{R}^3$. Any line or plane passing through the origin is a subspace of $\mathbb{R}^3$.

What about $W = \{(x,y,z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z = 0\}$



Consider the scalar $c=-1$ and vector $\vec{v}=(1,1,0)$. $\vec{v}\in W$, but $c\vec{v}=(-1,-1,0)\notin W$, so W is not a subspace of $\mathbb{R}^3$.

__Ex 2__ Consider the vector $\mathbb{R}^{3\times 3}$.
The space U of upper triangular matrices is a subspace of $\mathbb{R}^{3\times3}$.
Let $A, B \in U$ and $c\in\mathbb{R}$. Then

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad B=\begin{bmatrix} b_{11} & b_{21} & b_{31} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix}$$

$$(i)\ A+B=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ 0 & a_{22}+b_{22} & a_{23}+b_{23} \\ 0 & 0 & a_{33}+b_{33} \end{bmatrix}\in U$$

(ii) $cA = \begin{bmatrix} ca_{11} & ca_{12} & ca_{13} \\ 0 & ca_{22} & ca_{23} \\ 0 & 0 & ca_{33} \end{bmatrix} \in U$

<u>Ex 3</u> $Z = \{\vec{0}\}$ is a subspace of any vector space.

<u>The Column space of A</u>

We want to understand $A\vec{x} = \vec{b}$.
Recall that we can think of $A\vec{x}$ as a linear combination of the columns of A.

<u>Definition</u> The column space is the space formed by all linear combinations of the columns of a matrix A. We

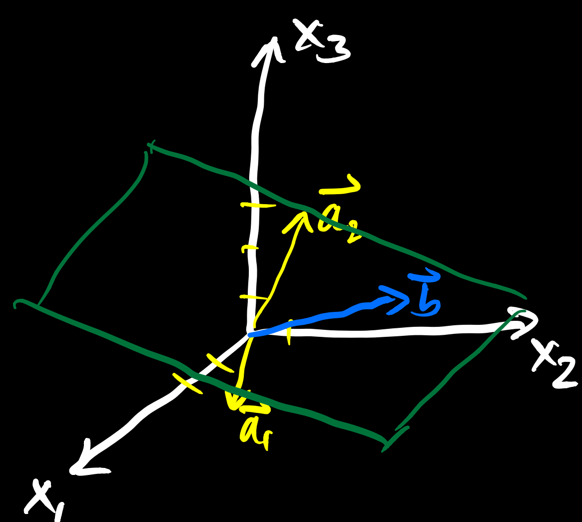
denote the column space of A as $C(A)$.

<u>Remark</u> To solve $A\vec{x} = \vec{b}$, we want to find $\vec{x} = (x_1, x_2, \dots, x_n)$ so that for $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$$

We see that $A\vec{x} = \vec{b}$ is solvable if and only if $\vec{b} \in C(A)$.

<u>Ex</u> $\begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 3 \end{bmatrix} \vec{x} = \vec{b}$



$x_1\vec{a}_1 + x_2\vec{a}_2$ fill a plane and a solution exists if and only if $\vec{b}$ lies in that plane.

<u>Definition</u> Let $S = \{\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_n\}$ be a collection of vectors in V. The span of S is the set of all linear combinations of the vectors in S, denoted $\mathrm{Span}(S)$.

<u>Remark</u> $\mathrm{Span}(S)$ is the smallest subspace containing S.

3.2 The Nullspace of A: Solving $A\vec{x} = \vec{0}$ and $R\vec{x} = 0$

We will examine solutions to the equation $A\vec{x} = \vec{0}$.

<u>Definition</u> The nullspace of A consists of vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$. We denote the nullspace of A by $N(A)$.

<u>Remark</u> The nullspace is a subspace of $\mathbb{R}^n$.

<u>Ex 1</u> Describe the nullspace of

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 12 \end{bmatrix}.$$

$A\vec{x} = 0$

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 4 & 12 & 0 \end{array}\right] r_2 - 4r_1$$

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Then $x_1 = -3x_2$. All points (x_1, x_2) on this line are in the nullspace of A!

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Ex 2 Describe the nullspace of

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

$$A\vec{x} = 0$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right] r_2 - 2r_1$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -3 & 0 \end{array}\right]$$

Then
$$x_1 + 2x_2 = 0$$
$$-3x_2 = 0$$

so $x_2 = 0$ and $x_1 = 0$.

Then $N(A) = Z = \{\vec{0}\}$. Here, A is nonsingular, i.e. invertible.

We see that we are using the tools we have developed so far to solve the equation $A\vec{x} = \vec{0}$.

The Reduced Row Echelon Form R

When A is not a square matrix, we will not arrive at an upper triangular matrix after elimination. Instead, we arrive at the reduced row echelon form $R = \text{rref}(A)$. The process is similar to Gauss-Jordon elimination!

1. Produce zeros above and below pivots
2. Produce ones in the pivots

Ex Describe the nullspace of

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 2 \\ 0 & 1 & 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1} & 0 & 2 & 1 & 0 & 2 \\ 0 & \boxed{1} & 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \vec{x} = \vec{0}$$

$$x_5 = -2x_6$$

$$x_2 = -2x_3 - x_4 - 2x_6$$

$$x_1 = -2x_3 - x_4 - 2x_6$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2x_3 - x_4 - 2x_6 \\ -2x_3 - x_4 - 2x_6 \\ x_3 \\ x_4 \\ -2x_6 \\ x_6 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -2 \\ -2 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

We have 3 free variables!

Notice number of pivots = number of free variables

Remark If $A \in \mathbb{R}^{m \times n}$ and $n > m$, then there is at least one free variable.

The Rank of a Matrix

Definition The rank of A is the number of pivots, denoted r.

Ex (Rank one matrix)

Let $\vec{u} = (1, 2, 3)$ and $\vec{v} = (1, 2, 3)$ and $A = uv^T$.

Then

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Let's find rref(A)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \begin{matrix} \\ r_2 - 2r_1 \\ r_3 - 3r_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We have only one pivot, so rank of A is 1.

Remark Later we will see that the rank of A, r, is the number of independent columns of A. This corresponds to the dimension of the column space, and $n-r$ is the dimension of the null space!