## Warm-up

For the following matrices, find the nullspace, determine the rank, and choose a set of columns that are linearly independent.

1. $$\begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 4 & 2 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$$

2. $$\begin{bmatrix} 1 & 2 \\ 4 & 1 \\ 2 & 2 \\ -1 & 1 \end{bmatrix}$$

1. $A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 4 & 2 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$

First we find rref$(A)$.

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 4 & 2 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ r_2 - r_1 \\ r_3 - 2r_1 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 5 & 2 & -1 \\ 0 & 3 & 0 & -2 \end{bmatrix} \begin{array}{l} r_1 + \frac{1}{5} r_2 \\ \\ r_3 - \frac{3}{5} r_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 2/5 & 4/5 \\ 0 & 5 & 2 & -1 \\ 0 & 0 & -6/5 & -7/5 \end{bmatrix} \begin{array}{l} \\ r_2/5 \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 2/5 & 4/5 \\ 0 & 1 & 2/5 & -1/5 \\ 0 & 0 & -6/5 & -7/5 \end{bmatrix} \begin{array}{l} r_1 + \frac{1}{3} r_3 \\ r_2 + \frac{1}{3} r_3 \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 5/15 \\ 0 & 1 & 0 & -10/15 \\ 0 & 0 & -6/5 & -7/5 \end{bmatrix} \; r_3 \cdot -\frac{5}{6}$$

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 1/3 \\ 0 & \boxed{1} & 0 & -2/3 \\ 0 & 0 & \boxed{1} & 7/6 \end{bmatrix} = \underline{\mathrm{rref}(A)}$$

$A\vec{x} = \vec{0}$

$\Downarrow$

$R\vec{x} = \vec{0}$

$$x_1 + \frac{1}{3}x_4 = 0 \implies x_1 = -\frac{1}{3}x_4$$

$$x_2 - \frac{2}{3}x_4 = 0 \qquad x_2 = \frac{2}{3}x_4$$

$$x_3 + \frac{7}{6}x_4 = 0 \qquad x_3 = -\frac{7}{6}x_4$$

$$\vec{x} = \begin{bmatrix} -\frac{1}{3}x_4 \\ \frac{2}{3}x_4 \\ -\frac{7}{6}x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -1/3 \\ 2/3 \\ -7/6 \\ 1 \end{bmatrix}$$

rank of $A$: $r = 3$

$1^{st}$ 3 columns of $A$ are linearly independent.

## 3.3 The Complete Solution to $A\vec{x} = \vec{b}$

In the previous section, we considered $A\vec{x} = \vec{0}$, and we found that the solution involved free variables. Now, we want to consider $A\vec{x} = \vec{b}$ where $\vec{b}$ is not $\vec{0}$.

### One particular solution $A\vec{x}_p = \vec{b}$

$\vec{x_p}$ is called the particular solution. When solving $A\vec{x} = \vec{b}$, the solution

$\vec{x}$ may involve free variables. When we take the free variables to be 0, we get $\vec{x}_p$.

Ex 1

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 1 & 2 & 1 & 4 & 6 \end{array}\right] r_3 - r_1$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & \boxed{1} & 2 & 2 \\ 0 & 0 & 1 & 3 & 5 \end{array}\right] r_3 - r_2$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} r_1 - r_3 \\ r_2 - 2r_3 \\ \phantom{0} \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

Now,

$$x_4 = 3$$

$$x_3 = -4$$

$$x_1 + 2x_2 = -2 \Rightarrow x_1 = -2 - 2x_2$$

$$\vec{x} = \begin{bmatrix} -2-2x_2 \\ x_2 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ -4 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Letting $x_2 = 0$, we find

$$\vec{x}_p = \begin{bmatrix} -2 \\ 0 \\ -4 \\ 3 \end{bmatrix}$$

Notice

$$\begin{bmatrix} \boxed{1} & 2 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix} \begin{bmatrix} \boxed{x_1} \\ x_2 \\ \boxed{x_3} \\ \boxed{x_4} \end{bmatrix} = \begin{bmatrix} \boxed{-2} \\ \boxed{-4} \\ \boxed{3} \end{bmatrix}$$

From the pivots

$$\vec{x}_p = \begin{bmatrix} -2 \\ 0 \\ -4 \\ 3 \end{bmatrix}$$

<u>Remarks</u> 1. The complete solution is

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

2. $\vec{x}_p$ solves $A\vec{x}_p = \vec{b}$

3. $x_{nullspace}$ solves $A\vec{x}_n = \vec{0}$

Notice

$$
\begin{aligned}
A\vec{x} &= A(\vec{x}_p + \vec{x}_n) \\
&= A\vec{x}_p + A\vec{x}_n \\
&= \vec{b} + \vec{0} \\
&= \vec{b}
\end{aligned}
$$

Recall that when we were finding the nullspace of $A$, the rank of $A$ is the number of pivots.

Full Column rank

A matrix $A$ has full column rank of $r = n$. Every matrix

A with full column rank has the following properties:

1. All columns of A are pivot columns
2. There are no free variables
3. The nullspace $N(A)$ contains only the zero vector.
4. If $A\vec{x} = \vec{b}$ has a solution, then it has only one solution.

Ex 2

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & -1 \\ 1 & 2 & 2 \\ -2 & -3 & -1 \end{array}\right] \begin{array}{l} \\ r_2 - r_1 \\ r_3 + 2r_1 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{array}\right] \begin{array}{l} \\ \\ r_3 + r_2 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} r_1 - r_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cc|c} \boxed{1} & 0 & -4 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & 0 \end{array}\right]$$

$A\vec{x} = 0$

$\Updownarrow$

$R\vec{x} = 0$

$\vec{x} = 0$

$$x_1 = -4$$
$$x_2 = 3$$
$$\vec{x} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

<u>Remark</u> 1. Notice $A$ has independent columns

2. With full column rank, $A\vec{x} = \vec{b}$ has one solution or no solution

<u>Full row rank</u>

A matrix has full row rank of $r = m$. Here, the rows are independent. The matrix in Example 1 had full row rank.

A matrix $A$ of full rank has the properties:

1. All rows have pivots, and $R$ has no zero rows
2. $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$
3. The column space is the whole space $\mathbb{R}^m$
4. There are $n - r = n - m$ free variables in the solution to $A\vec{x} = \vec{0}$.

The four possibilities for linear equations depend on the rank $r$

| | | |
|---|---|---|
| $r = m = n$ | $A\vec{x} = \vec{b}$ has 1 solution | $\text{rref}(A) = [I]$ |
| $r = m < n$ (full row rank) | $A\vec{x} = \vec{b}$ has infinitely many soln. | $\text{rref}(A) = [I \ F]$ |
| $r = n < m$ (full column rank) | $A\vec{x} = \vec{b}$ has 0 or 1 solution | $\text{rref}(A) = \begin{bmatrix} I \\ 0 \end{bmatrix}$ |
| $r < m$ and $r < n$ | $A\vec{x} = \vec{b}$ has 0 or infinitely many solutions | $\text{rref}(A) = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ |