3.4 Independence, Basis, and Dimension

We previously considered the column space of a matrix A. We want a way to talk about the size or dimension of a vector space. We can do so by considering a linearly independent set of vectors which span the vector space.

<u>Recall</u> 1. A collection of vectors $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent if

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$$

implies $x_1 = x_2 = \cdots = x_n = 0$.

2. The columns of A are

independent when $A\vec{x}=\vec{0}$ implies $\vec{x}=\vec{0}$.

Ex (a) $(1,0)$ and $(0,1)$ are independent

(b) $(1,1)$ and $(-1,-1)$ are dependent

(c) (a,b), (c,d), (e,f) are dependent

Last time, we said that the span of a collection of vectors is the set of all linear combinations. We want to know what the set of linear combinations fills.

Definition A set of vectors spans a space if their linear combinations fill the space.

Ex (a) $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

Looking at the set of linear combinations, we see

$$a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

so $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ spans $\mathbb{R}^3$ and is linearly independent.

(b) $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

spans $\mathbb{R}^3$ but is linearly dependent.

(c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ spans a plane and is linearly dependent.

Summary: We see that if we have too many vectors or not the "right" vectors, they are linearly dependent, and we may not have enough vectors to span the desired space.

A basis for a Vector Space

Definition Let V be a vector space. A basis for V is a linearly independent set of vectors in V which spans the space V.

<u>Remark</u> 1. Every vector in V is a unique combination of the basis vectors.

2. All bases for a vector space contain the same number of vectors.

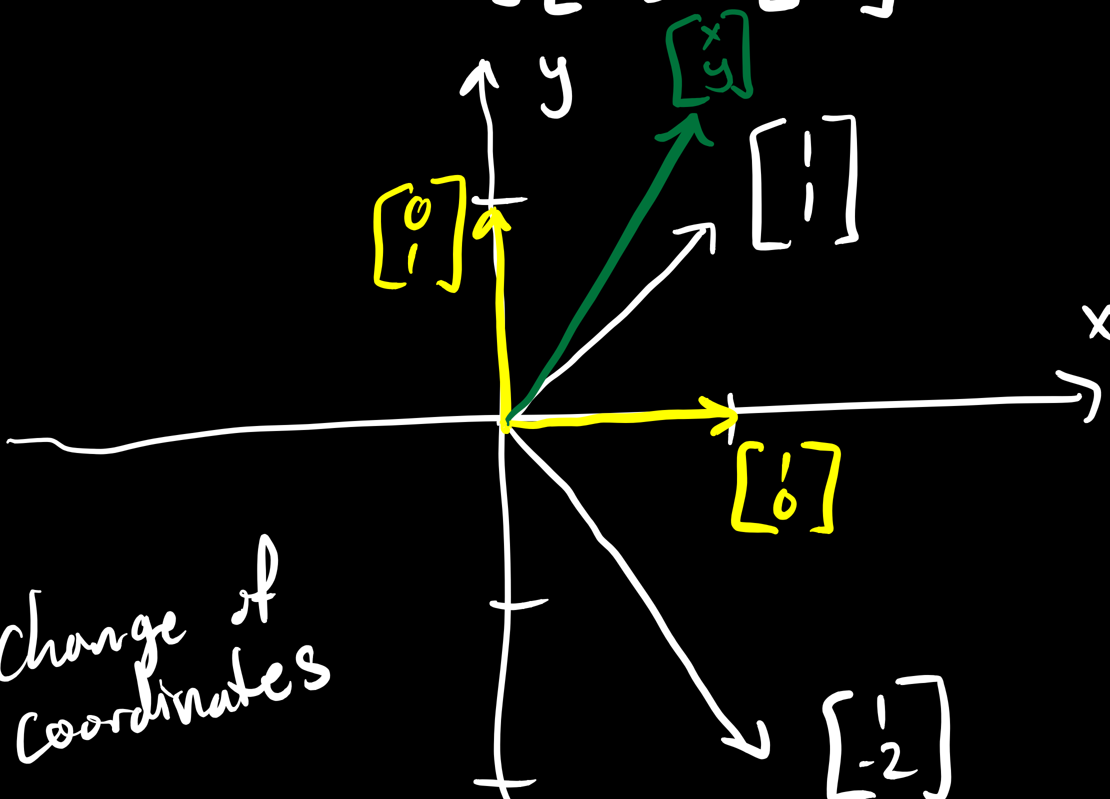
<u>Ex</u> (a) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is the Standard basis for $\mathbb{R}^2$.

Any vector in $\mathbb{R}^2$ may be written $x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

(b) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$ is also a basis for $\mathbb{R}^2$.

$$a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$



change of coordinates

The basis vectors are like building blocks of the vectors in the vector space. We see that there is a certain number of building blocks required.

<u>Definition</u> The dimension of a vector space is the number of vectors in every basis.

<u>Ex</u> (a) The dimension of $\mathbb{R}^2$ is 2
(b) The dimension of $\mathbb{R}^n$ is n

How does this all relate to matrices?

<u>Recall</u> A matrix A has full rank when $r = n$. Then there are n pivots and no free variables.

$\vec{x} = \vec{0}$ is the only element of the nullspace. The columns of A are independent.

Ex $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} = \vec{b} \overset{?}{=} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Column space is spanned by $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ which are linearly independent, so the column space is 2-dimensional. $\vec{b} \in \mathbb{R}^3$, but the column space is 2-dimens, so there are $\vec{b} \in \mathbb{R}^3$ for which there is no solution. For example

$$\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We have considered linear combinations of the columns, but what about the rows?

<u>Definition</u> The row space of a matrix is the subspace of $\mathbb{R}^n$ spanned by the rows. The row space of A is the column space of A^T.

<u>Ex</u> $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \vec{x} = \vec{b}$

The row space is spanned by

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$ Notice this matrix has full row rank.

<u>Note</u> If $A \in \mathbb{R}^{m \times n}$, the n columns are in $\mathbb{R}^m$ and span the column space. The m rows are in $\mathbb{R}^n$ and span the row space.

How can we find a basis for the column space?

The columns of an invertible $n \times n$ matrix are a basis for $\mathbb{R}^n$!

Why? Since A is invertible, $A\vec{x} = \vec{0}$

only has the solution $\vec{x}=\vec{0}$, so the columns of A are linearly independent.

Since A is invertible, $A\vec{x}=\vec{b}$ has a unique solution $\vec{x}=A^{-1}\vec{b}$ for any $\vec{b}\in\mathbb{R}^n$. Then the columns of A span $\mathbb{R}^n$.

What about other matrices?

The pivot columns of A are a basis for its column space. The pivot rows of A are a basis for its row space. So are the pivot rows of its echelon form R.

Ex Find a basis for

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} \right\}$$

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & 0 & 2 & 4 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

We find rref(A).

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & 0 & 2 & 4 \\ 0 & 1 & 1 & 2 \end{bmatrix} \begin{matrix} \\ r_2 - 2r_1 \\ \\ \end{matrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & 1 & 2 \end{bmatrix} \begin{matrix} r_1 + \frac{1}{2} r_2 \\ \\ r_3 - \frac{1}{2} r_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} r_2/2$$

$$= \begin{bmatrix} \boxed{1} & 0 & 1 & 2 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Columns 1 and 2 are our pivot columns so $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

forms a basis of the collection of vectors.

$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 4 \end{bmatrix} \right\}$ forms a basis of the row space of A. Also,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$ forms a basis

of the row space.

<u>Remark</u> The rank of a matrix is the dimension of the column space.

<u>Bases for Matrix Spaces and Function Spaces</u>

<u>Matrix Spaces</u> We previously mentioned that $\mathbb{R}^{m \times n}$ had dimension mn. What would be a basis?

Let's consider $\mathbb{R}^{2\times 2}$. A basis for $\mathbb{R}^{2\times 2}$ is

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{A_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{A_3}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{A_4}$$

We see that any $A \in \mathbb{R}^{2\times 2}$ may be written as

$$c_1 A_1 + c_2 A_2 + c_3 A_3 + c_4 A_4 = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$$

Notice that $\{A_1, A_2, A_4\}$ forms a basis for the space of 2×2 upper triangular matrices.

Function Spaces

Consider the differential equation $y''=0$. It is solved by any linear function $y=cx+d$. Notice that $cx+d$ is in the nullspace of the second derivative!

1 and x for the basis of the space of solutions to $y''=0$.
The solution space is 2-dimensional.

Consider $y''=2$. A particular solution is $y=x^2$. The complete solution is $y = x^2 + cx + d$. Notice

$$y = y_p + y_{nullspace}!$$

A linear differential equation is like the linear matrix equation $A\vec{x} = \vec{b}$, but we used Calculus instead of linear algebra!