3.5 Dimensions of the Four Subspaces

Last time We introduced the row space and the idea of a basis and dimension.

We have seen how the size of a matrix and its rank determines the number of solutions to $A\vec{x} = \vec{b}$. Today, we want to connect the row space, column space, nullspace, left nullspace and their dimension.

Four Fundamental Subspaces

Let A be an $m \times n$ matrix. $\begin{array}{cc} & n \\ m & \begin{bmatrix} \quad & \quad \end{bmatrix} \end{array}$

1. The row space is $C(A^T)$, a subspace of $\mathbb{R}^n$

2. The column space is $C(A)$, a subspace of $\mathbb{R}^m$

3. The nullspace is $N(A)$, a subspace of $\mathbb{R}^n$

4. The left nullspace is $N(A^T)$, a subspace of $\mathbb{R}^m$.

<u>Definition</u> The left nullspace is the vector space spanned by solutions to $y^T A = 0^T$

$$[0 \cdots 0] = [y_1 \cdots y_m] \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & \ddots & \\ \vdots & & a_{mn} \end{bmatrix}$$

<u>Remark</u> 1. To find the left nullspace, we solve

$$y^T A = 0^T$$

$$(y^T A)^T = (0^T)^T$$

$$A^T y = 0$$

i.e. $N(A^T)$ is the left nullspace

2. The nullity of A is the dimension of the nullspace.

<u>Ex</u> Find a basis for each of the four fundamental spaces of A and determine the dimension of each space.

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 2 & 3 \\ 0 & 0 & 2 & 0 & 2 \end{bmatrix}$$

First we find $\text{rref}(A)$ and $\text{rref}(A^T)$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 2 & 3 \\ 0 & 0 & 2 & 0 & 2 \end{bmatrix} r_2 - 2r_1$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 2 \end{bmatrix} r_3 - 2r_2$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \operatorname{rref}(A)$$

1. The pivot rows of $\operatorname{rref}(A)$ form a basis of $C(A^T)$

$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$ form a basis

of $C(A^T)$, so the row space

has dimension 2.

2. The pivot columns of A form a basis of C(A).

$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ form a basis of C(A), and C(A) is 2-dimensional.

Notice C(A) and $C(A^T)$ have the same dimension! The dimension is 2 = rank(A).

3. To find a basis of N(A), we solve $R\vec{x} = \vec{0}$.

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \vec{0}$$

$$x_1 = -2x_2 - x_4 - x_5$$

$$x_3 = -x_5$$

$$\vec{x} = \begin{bmatrix} -2x_2 - x_4 - x_5 \\ x_2 \\ -x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

So $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ forms a basis of $N(A)$ and $\text{nullity}(A) = 3$.

Notice that

$\text{nullity}(A) + \text{rank}(A) = 3 + 2 = 5 =$ number of Columns

4. To find a basis of the left nullspace of A, we solve

$$x^T R = 0^T \quad \text{or} \quad R^T x = 0.$$

$x^T R = 0^T$ looks for a linear combination of rows that produce 0^T.

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \vec{0}$$

$$x_1 = 0$$
$$2x_1 = 0$$
$$x_2 = 0$$
$$x_1 = 0$$
$$x_1 + x_2 = 0$$

Then $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

So $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ forms a basis for $N(A^T)$ and $\text{nullity}(A) = 1$.

Again, notice that

$$\begin{aligned} \text{rank}(A^T) + \text{nullity}(A^T) &= 2 + 1 \\ &= 3 \\ &= \text{number of rows of } A \end{aligned}$$

Now, we look at 2 fundamental theorems of linear algebra.

<u>Theorem</u> Let A be an $m \times n$ matrix with $\text{rank}(A) = r$. The column space and row space of A have the same dimension, $\text{rank}(A)$.

Reason: The rank(A) is the number of pivots in rref(A). The basis vectors of C(A) are determined by the pivot columns. The basis vectors of $C(A^T)$ are determined by the pivot rows of rref(A). The number of basis vectors is equal to the number of pivots, which is the rank(A), so the column space and row space of A have the same dimension.

Theorem (Rank-Nullity) Let A be an $m \times n$ matrix. Then

$$\text{rank}(A) + \text{nullity}(A) = n.$$

<u>Proof</u> Suppose nullity $(A) = K$. Let $\{\vec{v}_1, \ldots, \vec{v}_K\}$ be a basis for $N(A)$. There are vectors $\{\vec{v}_{K+1}, \ldots, \vec{v}_n\}$ so that $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$. We will show that $\{A\vec{v}_{K+1}, \ldots, A\vec{v}_n\}$ is a basis of $C(A)$. $A\vec{v}_1, \ldots, A\vec{v}_n$ spans the column space, and since $A\vec{v}_i = 0$ for $i \leq K$, $A\vec{v}_{K+1}, \ldots A\vec{v}_n$ spans the column space. Now, we want to show that $A\vec{v}_{K+1}, \ldots, A\vec{v}_n$ are linearly independent. Suppose we can find scalars c_i so that

$$\sum_{i=k+1}^{n} c_i (A\vec{v}_{i+1}) = 0$$

then

$$A\left(\sum_{i=k+1}^{n} c_i \vec{v}_{i+1}\right) = 0$$

So the vector $\sum_{i=k+1}^{n} c_i \vec{v}_{i+1}$ is in the nullspace of A. Since $\vec{v}_1, \ldots, \vec{v}_k$ forms a basis of $N(A)$, we can find scalars b_i so that

$$\sum_{i=k+1}^{n} c_i \vec{v}_{i+1} = \sum_{i=1}^{k} b_i \vec{v}_i$$

Then

$$\sum_{i=k+1}^{n} c_i \vec{v}_{i+1} - \sum_{i=1}^{k} b_i \vec{v}_i = 0.$$

Since $\vec{v}_1, \ldots, \vec{v}_n$ is a basis of $\mathbb{R}^n$, $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent.

so

$$b_1 = \cdots = b_k = c_{k+1} = \cdots = c_n = 0.$$

Thus

$$\sum_{i=k+1}^{n} c_i (A\vec{v}_i) = 0$$

means $c_{k+1} = \cdots = c_n = 0$, so

$A\vec{v}_{k+1}, \ldots, A\vec{v}_n$ are linearly independent.

Ergo, $A\vec{v}_{k+1}, \ldots, A\vec{v}_n$ form a basis of $C(A)$.

We see that $\text{rank}(A) = n-k$ and $\text{nullity}(A) = k$, so

$$\begin{aligned} \text{rank}(A) + \text{nullity}(A) &= n-k+k \\ &= n. \end{aligned}$$

$\square$

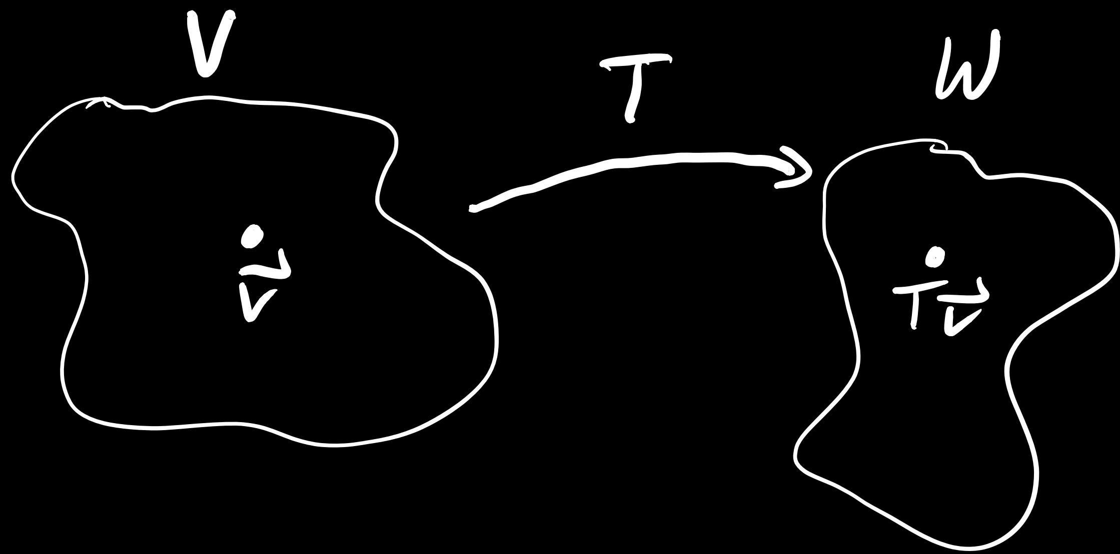
Linear Transformation

We have looked at matrices acting on vectors to produce new vectors. If A is an $m \times n$ matrix, it takes an n-dimensional vector $\vec{x}$ as an input and produces an m-dimensional vector $\vec{b}$ through $A\vec{x} = \vec{b}$. We can generalize this idea to a linear transformation.

Definition Let V and W be vector spaces over the field F. A linear transformation from V into W is a function T from V into W such that

$$T(a\vec{v}+\vec{b}) = aT\vec{v} + T\vec{w}$$

for all $a \in F$ and $\vec{v}, \vec{w} \in V$.



Remarks 1. We see that A is a linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$.

2. The idea of rank, nullity, nullspace, and column space generalize to the linear transformation viewpoint.

<u>Ex 1</u> Let A be an $m\times n$ matrix. Then $T(X)=AX$ is a linear transformation from $\mathbb{R}^{n\times 1}$ into $\mathbb{R}^{m\times 1}$.

<u>Ex 2</u> Let P be an $m\times m$ matrix and Q an $n\times n$ matrix. Then

$$T(A) = PAQ$$

is a linear transformation from $F^{m\times n}$ into $F^{m\times n}$.

$$\begin{aligned} T(cA+B) &= P(cA+B)Q \\ &= cPAQ + PBQ \\ &= cT(A) + T(B). \end{aligned}$$

<u>Ex 3</u> Let V be the space of continuous real-valued functions, and

$$(Tf)(x) = \int_0^x f(t)\,dt$$

Then T is a linear transformation from V into V.

<u>Ex 4</u> Let V be the space of polynomial functions

$$f(x) = c_0 + c_1 x + \cdots + c_k x^k .$$

Let

$$(Df)(x) = c_1 + 2c_2 x + \cdots + k c_k x^{k-1}.$$

Then D is a linear transformation from V into V. This is differentiation.