MAT 22A Problem Set 1 (Due 6/26 8 AM)

1. Let
$$\mathbf{v} = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix}2\\2\\0\end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix}2\\0\\3\end{bmatrix}.$$
Describe the following linear combinations geometrically where $a, b, c \in \mathbb{R}$. (Lie on a line, lie in a plane, etc.)
   (a) $a\mathbf{v}$
   (b) $a\mathbf{v} + b\mathbf{w}$
   (c) $a\mathbf{v} + b\mathbf{w} + c\mathbf{u}$
   (d) $a\mathbf{v} + c\mathbf{u}$
   (e) $a\mathbf{v} + b\mathbf{w} + c\mathbf{u}$

2. Draw the following vectors in the xy-plane.
   (a) $\mathbf{v} = (2, 1)$
   (b) $\mathbf{w} = (-1, 1)$
   (c) $\mathbf{v} + \mathbf{w}$
   (d) $\mathbf{v} - \mathbf{w}$

3. Let
$$\mathbf{v} = \begin{bmatrix}1\\2\\1\end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix}2\\2\\2\end{bmatrix}.$$
Do the vectors $\mathbf{v}$, $\mathbf{w}$ lie in a plane? If so, what is the equation of the plane? If not, find the vector equation of the line that passes through the origin and points in the same direction as $\mathbf{v}$ and $\mathbf{w}$.

4. Consider the linear combination
$$a\begin{bmatrix}2\\1\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}$$
where $0 \le a \le 4$ and $-1 \le b \le 0$. Shade the region covered by the linear combination.

5. If possible, find a and b so that
$$a\mathbf{v} + b\mathbf{w} = \mathbf{c}$$
where
   (a)
$$\mathbf{v} = \begin{bmatrix}1\\1\end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix}2\\1\end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix}-1\\1\end{bmatrix}$$

(b)

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

6. Let

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

   Compute the following

   (a) $\mathbf{v} \cdot \mathbf{w}$

   (b) $\mathbf{v} \cdot (\mathbf{w} + \mathbf{u})$

   (c) Check the Schwarz inequality $|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\|\|\mathbf{w}\|$.

7. Let $\mathbf{v} \in \mathbb{R}^{2n}$ such that $v_i = 1$ for $i = 1, 2, \ldots, 2n$. Find a unit vector $\mathbf{u}$ that points in the same direction as $\mathbf{v}$. Find a vector $\mathbf{w}$ that is perpendicular to $\mathbf{v}$.

8. Prove

$$\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 = 2\|\mathbf{v}\|^2 + 2\|\mathbf{w}\|^2.$$

9. Prove

$$\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\| \cos\theta + \|\mathbf{w}\|^2.$$

10. Let $\mathbf{w} = (a, b, c)$ and $\mathbf{v} = (c, b, a)$ where $a + b + c = 0$. Find the angle between $\mathbf{w}$ and $\mathbf{v}$.