MAT 22A Problem Set 10 Solution

1. Consider the plane $x + y + z = 0$ as a subspace of $\mathbb{R}^3$. Choose two orthogonal vectors and make them orthonormal. Finally, find the projection matrix and projection of $\mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ on the plane. Make a rough sketch of the plane, $\mathbf{b}$, and the projection of $\mathbf{b}$ onto the plane.

   Solution.

   We see that $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ are both in the plane $x+y+z = 0$, and clearly $\mathbf{v}^{\mathrm{T}}\mathbf{w} = 0$. Now, $\|\mathbf{v}\| = \|\mathbf{w}\| = \sqrt{2}$, so $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$ form an orthonormal basis of the plane. Let

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{bmatrix}.$$

   Since the columns of A are orthonormal, the projection matrix is $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = AA^{\mathrm{T}}$. We find that

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   and so the projection of $\mathbf{b}$ onto the plane is

$$\mathbf{p} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix}.$$

   □

2. Find an orthonormal basis of the following vector space S.

$$S = \operatorname{span}\left(\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 2 \\ 3 \end{bmatrix} \right)$$

   Solution.

   First, we see that

$$\begin{vmatrix} 2 & 1 & 2 & 5 \\ 2 & 3 & 2 & 7 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{vmatrix} = 0$$

so the columns are linearly dependent. Now, row-reducing the matrix, we have

$$\begin{bmatrix} 2 & 1 & 2 & 5 \\ 2 & 3 & 2 & 7 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{matrix} \\ r_2 - r_1 \\ r_3 - \frac{1}{2}r_1 \\ \\ \end{matrix} = \begin{bmatrix} 2 & 1 & 2 & 5 \\ 0 & 2 & 0 & 2 \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{matrix} r_1 - \frac{1}{2}r_2 \\ \\ r_3 + \frac{1}{4}r_2 \\ r_4 - \frac{1}{2}r_2 \end{matrix}$$

$$= \begin{bmatrix} 2 & 0 & 2 & 4 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} \begin{matrix} r_1 - r_4 \\ \\ \\ \\ \end{matrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and so we see that $\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$ form a basis of S. Now, we use Gram-Schmidt to form an orthonormal basis. First, we orthogonalize.

$$\mathbf{w}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{w}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}}{\left\| \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\|^2} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix} - \frac{8}{9} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{7}{9} \\ \frac{11}{9} \\ -\frac{8}{9} \\ 1 \end{bmatrix}$$

$$\mathbf{w}_3 = \begin{bmatrix}2\\2\\1\\2\end{bmatrix} - \frac{\begin{bmatrix}2 & 2 & 1 & 2\end{bmatrix}\begin{bmatrix}-\frac{7}{9}\\\frac{11}{9}\\-\frac{8}{9}\\1\end{bmatrix}}{\left\|\begin{bmatrix}-\frac{7}{9}\\\frac{11}{9}\\-\frac{8}{9}\\1\end{bmatrix}\right\|^2}\begin{bmatrix}-\frac{7}{9}\\\frac{11}{9}\\-\frac{8}{9}\\1\end{bmatrix} - \frac{\begin{bmatrix}2 & 2 & 1 & 2\end{bmatrix}\begin{bmatrix}2\\2\\1\\0\end{bmatrix}}{\left\|\begin{bmatrix}2\\2\\1\\0\end{bmatrix}\right\|^2}\begin{bmatrix}2\\2\\1\\0\end{bmatrix}$$

$$= \begin{bmatrix}2\\2\\1\\2\end{bmatrix} - \frac{18}{35}\begin{bmatrix}-\frac{7}{9}\\\frac{11}{9}\\-\frac{8}{9}\\1\end{bmatrix} - \begin{bmatrix}2\\2\\1\\0\end{bmatrix}$$

$$= \begin{bmatrix}0\\0\\0\\2\end{bmatrix} - \frac{18}{35}\begin{bmatrix}-\frac{7}{9}\\\frac{11}{9}\\-\frac{8}{9}\\1\end{bmatrix}$$

$$= \begin{bmatrix}\frac{2}{5}\\-\frac{22}{35}\\\frac{16}{35}\\\frac{52}{35}\end{bmatrix}$$

Now, we normalize

$$\mathbf{v}_1 = \frac{1}{3}\begin{bmatrix}2\\2\\1\\0\end{bmatrix}$$

$$\mathbf{v}_2 = \frac{3}{\sqrt{35}}\begin{bmatrix}-\frac{7}{9}\\\frac{11}{9}\\-\frac{8}{9}\\1\end{bmatrix}$$

$$\mathbf{v}_3 = \sqrt{\frac{35}{104}}\begin{bmatrix}\frac{2}{5}\\-\frac{22}{35}\\\frac{16}{35}\\\frac{52}{35}\end{bmatrix}$$

and so $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ forms an orthonormal basis of S. □

3. Suppose the columns of an $m \times n$ matrix A are orthogonal. Consider the system $A\mathbf{x} = \mathbf{b}$. Find an expression for x_i (the i^{th} entry of $\mathbf{x}$).

   Solution.

   Let the columns of A be $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Multiplyting A^{T} to each side of

$A\mathbf{x} = \mathbf{b}$, we get

$$\begin{bmatrix} \|\mathbf{v}_1\|^2 & 0 & \cdots & 0 \\ 0 & \|\mathbf{v}_2\|^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \|\mathbf{v}_n\|^2 \end{bmatrix} \mathbf{x} = A^{\mathrm{T}}\mathbf{b}.$$

We see that $\|\mathbf{v}_i\|^2 x_i = \mathbf{v}_i^{\mathrm{T}}\mathbf{b}$, so $x_i = \frac{\mathbf{v}_i^{\mathrm{T}}\mathbf{b}}{\|\mathbf{v}_i\|^2}$. □

4. Consider the following system $A\mathbf{x} = \mathbf{b}$. Find the $A = QR$ decomposition of the following matrix. Then solve the system using the decomposition.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Solution.

To find Q, we perform Gram-Schmidt on the columns of A. First we orthogonalize.

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{w}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\|^2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{aligned} \mathbf{w}_3 &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\|^2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\|^2} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \end{aligned}$$

Now, we normalize

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\mathbf{v}_3 = \sqrt{\frac{3}{2}} \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

□

We see that

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{3}\sqrt{\frac{3}{2}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{3}\sqrt{\frac{3}{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{3}\sqrt{\frac{3}{2}} \end{bmatrix}.$$

Since Q is orthogonal, $Q^{\mathrm{T}}Q = I$, so $R = Q^{\mathrm{T}}A$.

$$\begin{aligned} R &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{3}\sqrt{\frac{3}{2}} & \frac{2}{3}\sqrt{\frac{3}{2}} & \frac{1}{3}\sqrt{\frac{3}{2}} \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{2}{3}\sqrt{\frac{3}{2}} \end{bmatrix}. \end{aligned}$$

Now, we have $QR\mathbf{x} = \mathbf{b}$, so $R\mathbf{x} = Q^{\mathrm{T}}\mathbf{b}$.

$$\begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{2}{3}\sqrt{\frac{3}{2}} \end{bmatrix} \mathbf{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{3}\sqrt{\frac{3}{2}} & \frac{2}{3}\sqrt{\frac{3}{2}} & \frac{1}{3}\sqrt{\frac{3}{2}} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} \\ \frac{2}{3}\sqrt{\frac{3}{2}} \end{bmatrix}.$$

Now, using back substitution, we find that

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$