MAT 22A Problem Set 11 Solutions

1. Compute the determinant of A.

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 2 & 1 \\ 1 & 0 & -1 & 0 \\ -2 & 1 & 0 & 0 \end{bmatrix}.$$

Solution.

We compute the determinant of A using the cofactor method, and going down the 4^{th} column.

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & 2 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{vmatrix} \\ &= \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} - 2 \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} \\ &= 1 - 2(0-2) - 1 \\ &= 4 \end{aligned}$$

□

2. Find the eigenvalues and eigenvectors of A.

$$A = \begin{bmatrix} -3 & -4 & -5 & -1 \\ 4 & 4 & 4 & 4 \\ -4 & -2 & -2 & -4 \\ 3 & 6 & 3 & 1 \end{bmatrix}.$$

Solution.

First, we find the eigenvalues of A

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -3-\lambda & -4 & -5 & -1 \\ 4 & 4-\lambda & 4 & 4 \\ -4 & -2 & -2-\lambda & -4 \\ 3 & 6 & 3 & 1-\lambda \end{vmatrix} \\ &= (-3-\lambda) \underbrace{\begin{vmatrix} 4-\lambda & 4 & 4 \\ -2 & -2-\lambda & -4 \\ 6 & 3 & 1-\lambda \end{vmatrix}}_{1} + 4 \underbrace{\begin{vmatrix} 4 & 4 & 4 \\ -4 & -2-\lambda & -4 \\ 3 & 3 & 1-\lambda \end{vmatrix}}_{2} \\ &\quad - 5 \underbrace{\begin{vmatrix} 4 & 4-\lambda & 4 \\ -4 & -2 & -4 \\ 3 & 6 & 1-\lambda \end{vmatrix}}_{3} + \underbrace{\begin{vmatrix} 4 & 4-\lambda & 4 \\ -4 & -2 & -2-\lambda \\ 3 & 6 & 3 \end{vmatrix}}_{4} \end{aligned}$$

First,

$$\textcircled{1} = \begin{vmatrix} 4-\lambda & 4 & 4 \\ -2 & -2-\lambda & -4 \\ 6 & 3 & 1-\lambda \end{vmatrix}$$
$$\begin{aligned} &= (4-\lambda)\left[(-2-\lambda)(1-\lambda)+12\right] - 4\left[-2(1-\lambda)+24\right] + 4\left[-6-6(-2-\lambda)\right] \\ &= (4-\lambda)(\lambda^2+\lambda+10) - 8(\lambda+11) + 24(\lambda+1) \\ &= -\lambda^3 + 3\lambda^2 + 10\lambda - 24. \end{aligned}$$

Second,

$$\textcircled{2} = \begin{vmatrix} 4 & 4 & 4 \\ -4 & -2-\lambda & -4 \\ 3 & 3 & 1-\lambda \end{vmatrix}$$
$$\begin{aligned} &= 4\left[(-2-\lambda)(1-\lambda)+12\right] - 4\left[-4(1-\lambda)+12\right] + 4\left[-12-3(-2-\lambda)\right] \\ &= 4(\lambda^2+\lambda+10) - 16(\lambda+2) + 12(\lambda-2) \\ &= 4\lambda^2 - 16. \end{aligned}$$

Third,

$$\textcircled{3} = \begin{vmatrix} 4 & 4-\lambda & 4 \\ -4 & -2 & -4 \\ 3 & 6 & 1-\lambda \end{vmatrix}$$
$$\begin{aligned} &= 4\left[-2(1-\lambda)+24\right] - (4-\lambda)\left[-4(1-\lambda)+12\right] + 4\left[-24+6\right] \\ &= 8(\lambda+11) - (4-\lambda)4(\lambda+2) - 72 \\ &= 4\lambda^2 - 16. \end{aligned}$$

Fourth,

$$\textcircled{4} = \begin{vmatrix} 4 & 4-\lambda & 4 \\ -4 & -2 & -2-\lambda \\ 3 & 6 & 3 \end{vmatrix}$$
$$\begin{aligned} &= 4\left[-6-6(-2-\lambda)\right] - (4-\lambda)\left[-12-3(-2-\lambda)\right] + 4\left[-24+6\right] \\ &= 24(\lambda+1) - (4-\lambda)3(\lambda-2) - 72 \\ &= 3\lambda^2 + 6\lambda - 24. \end{aligned}$$

Now,

$$\begin{aligned} \det(A) &= (-3-\lambda)(-\lambda^3+3\lambda^2+10\lambda-24) + 4(4\lambda^2-16) - 5(4\lambda^2-16) + (3\lambda^2+6\lambda-24) \\ &= (-3-\lambda)(-\lambda^3+3\lambda^2+10\lambda-24) - 3\lambda^2 + 6\lambda - 8 \\ &= \lambda^4 - 20\lambda^2 + 64. \end{aligned}$$

Let $p(\lambda) = \lambda^4 - 20\lambda^2 + 64$. By either graphing the polynomial, trying some random guesses, or using some other means, we find that $\lambda = 2$ and

$\lambda = -2$ are both roots of $p(\lambda)$. This means that $(\lambda-2)(\lambda+2) = \lambda^2 - 4$ divides $p(\lambda)$. Now, $\frac{p(\lambda)}{\lambda^2-4} = \lambda^2 - 16$ so we find that $\lambda = 4$ and $\lambda = -4$ are also eigenvalues of A.

Now, we find the eigenvectors of A. First, let's consider $\lambda_1 = 2$. Then

$$(A - 2I)\mathbf{v}_1 = \mathbf{0}$$

$$\begin{vmatrix} -5 & -4 & -5 & -1 \\ 4 & 2 & 4 & 4 \\ -4 & -2 & -4 & -4 \\ 3 & 6 & 3 & -1 \end{vmatrix} \mathbf{v}_1 = \mathbf{0}$$

Now, after row reducing, we find that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$. Next, we consider $\lambda_2 = -2$. Then

$$(A + 2I)\mathbf{v}_2 = \mathbf{0}$$

$$\begin{vmatrix} -1 & -4 & -5 & -1 \\ 4 & 6 & 4 & 4 \\ -4 & -2 & 0 & -4 \\ 3 & 6 & 3 & 3 \end{vmatrix} \mathbf{v}_2 = \mathbf{0}$$

After row reducing, we find that $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$. Now, let's consider $\lambda_3 = 4$. Then

$$(A - 4I)\mathbf{v}_3 = \mathbf{0}$$

$$\begin{vmatrix} -7 & -4 & -5 & -1 \\ 4 & 0 & 4 & 4 \\ -4 & -2 & -6 & -4 \\ 3 & 6 & 3 & -3 \end{vmatrix} \mathbf{v}_3 = \mathbf{0}$$

After row reducing, we find that $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$. Finally, let's consider $\lambda_4 = -4$. Then

$$(A + 4I)\mathbf{v}_4 = \mathbf{0}$$

$$\begin{vmatrix} 1 & -4 & -5 & -1 \\ 4 & 8 & 4 & 4 \\ -4 & -2 & 2 & -4 \\ 3 & 6 & 3 & 5 \end{vmatrix} \mathbf{v}_4 = \mathbf{0}$$

After row reducing, we find that $\mathbf{v}_4 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$. $\square$

3. Find the $A = X\Lambda X^{-1}$ decomposition of A from problem 2. X is a matrix whose columns are the eigenvectors of A and Λ is a diagonal matrix whose entries are the eigenvalues of A. Next, compute the matrix products to show that indeed $A = X\Lambda X^{-1}$. Determine the eigenvalues of A^k and diagonalize A^k.

   Solution.

   In problem 2, we found the eigenvalues and eigenvectors of A. We then have that

$$X = \begin{bmatrix} -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

   and

$$\Lambda = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

   $\square$

   Computing X^{-1}, we find that

$$\begin{aligned} A &= \begin{bmatrix} -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -0.5 & 0 & -0.5 & -0.5 \\ 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 & -0.5 \\ 0 & -1 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 & 2 \\ 2 & 4 & 2 & 2 \\ -1 & -2 & -1 & 1 \\ 0 & -2 & -2 & 0 \end{bmatrix} \\ &= A \end{aligned}$$

   The eigenvalues of A are 2, -2, 4, and -4, so the eigenvalues of A^k are 2^k, $(-2)^k$, 4^k, and $(-4)^k$. The diagonalization of A^k is

$$\Lambda = X^{-1}\Lambda^k X.$$