MAT 22A Problem Set 1 (Due 6/26 8 AM)

1. Let
$$\mathbf{v} = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix}2\\2\\0\end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix}2\\0\\3\end{bmatrix}.$$

   Describe the following linear combinations geometrically where $a, b, c \in \mathbb{R}$. (Lie on a line, lie in a plane, etc.)

   (a) $a\mathbf{v}$
   (b) $a\mathbf{v} + b\mathbf{w}$
   (c) $a\mathbf{v} + b\mathbf{w} + c\mathbf{u}$
   (d) $a\mathbf{v} + c\mathbf{u}$
   (e) $a\mathbf{v} + b\mathbf{w} + c\mathbf{u}$

   Solution.

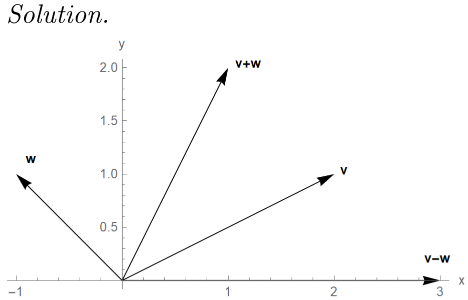
   (a) Lie on a line
   (b) Lie on a line
   (c) Lie in a plane
   (d) Lie in a a plane
   (e) Repeat of (c)

   □

2. Draw the following vectors in the xy-plane.

   (a) $\mathbf{v} = (2, 1)$
   (b) $\mathbf{w} = (-1, 1)$
   (c) $\mathbf{v} + \mathbf{w}$
   (d) $\mathbf{v} - \mathbf{w}$

   Solution.



   □

3. Let

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

Do the vectors $\mathbf{v}$, $\mathbf{w}$ lie in a plane? If so, what is the equation of the plane? If not, find the vector equation of the line that passes through the origin and points in the same direction as $\mathbf{v}$ and $\mathbf{w}$.

Solution.

Since $\mathbf{v} \neq c\mathbf{w}$ for some scalar c, $\mathbf{v}$ and $\mathbf{w}$ do not lie on a line. Therefore $\mathbf{v}$ and $\mathbf{w}$ lie in a plane. Recall the equation of a plane

$$\mathbf{n} \cdot (x - p_1, x - p_2, x - p_3) = 0$$

where $\mathbf{n}$ is a vector normal to the plane and $P = (p_1, p_2, p_3)$ is a point on the plane. The plane passes through the origin, so we may take $P = (0, 0, 0)$. A vector normal to the plane can be found by computing the cross product of $\mathbf{v}$ and $\mathbf{w}$,

$$\mathbf{v} \times \mathbf{w} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}.$$

Therefore, we find that the equation of the plane is

$$2x - 2z = 0.$$

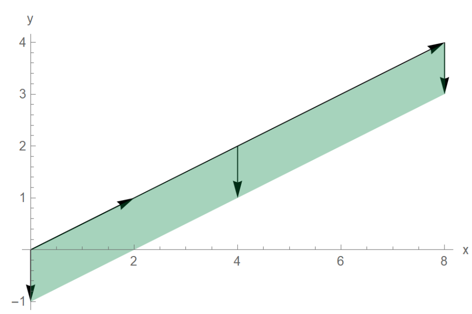
□

4. Consider the linear combination

$$a \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where $0 \leq a \leq 4$ and $-1 \leq b \leq 0$. Shade the region covered by the linear combination.

Solution.



□

5. If possible, find a and b so that

$$a\mathbf{v} + b\mathbf{w} = \mathbf{c}$$

where

(a)
$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

(b)
$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Solution.

(a) We have the equations $a + 2b = -1$ and $a + b = 1$. From the second equation, we find $b = 1 - a$. Plugging the expression for b into the first equation, we find $a = 3$. Then $b = -2$.

(b) Since the thrid entry of $\mathbf{c}$ is 0, we see that we must require $b = 0$. Then, we have that $a = -1$ and $a = 1$. We cann find a and b to form the desired linear combination!

□

6. Let
$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

Compute the following

(a) $\mathbf{v} \cdot \mathbf{w}$

(b) $\mathbf{v} \cdot (\mathbf{w} + \mathbf{u})$

(c) Check the Schwarz inequality $|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\|\|\mathbf{w}\|$.

Solution.

(a) $\mathbf{v} \cdot \mathbf{w} = 2(1) + 1(0) + 1(1) = 3$

(b) $\mathbf{v} \cdot (\mathbf{w} + \mathbf{u}) = 2(1 - 1) + 1(0 + 1) + 1(1 + 1) = 3$

(c) $\|\mathbf{v}\| = \sqrt{6}$, $\|\mathbf{w}\| = \sqrt{2}$ and $\mathbf{v} \cdot \mathbf{w} = 3 = \sqrt{9}$. $\sqrt{9} < \sqrt{12}$, so we see that indeed the Schwarz inequality holds.

□

7. Let $\mathbf{v} \in \mathbb{R}^{2n}$ such that $v_i = 1$ for $i = 1, 2, \ldots, 2n$. Find a unit vector $\mathbf{u}$ that points in the same direction as $\mathbf{v}$. Find a vector $\mathbf{w}$ that is perpendicular to $\mathbf{v}$.

Solution.

We can find $\mathbf{u}$ by scaling $\mathbf{v}$ by $\frac{1}{\|v\|}$. We have

$$\|v\| = \sqrt{\sum_{i=1}^{2n} v_i^2} = \sqrt{\sum_{i=1}^{2n} 1} = \sqrt{2n}$$

so

$$\mathbf{u} = \frac{1}{\sqrt{2n}}\mathbf{v}.$$

To find a vector $\mathbf{w}$ that is perpendicular to $\mathbf{v}$, we want to find $\mathbf{w}$ so that $\mathbf{w} \cdot \mathbf{v} = 0$. Now,

$$\begin{aligned} \mathbf{w} \cdot \mathbf{v} &= \sum_{i=1}^{2n} w_i v_i \\ &= \sum_{i=1}^{2n} w_i \end{aligned}$$

so we see that we need the entries of $\mathbf{w}$ to sum to zero. There are many vectors that would work such as

$$\begin{bmatrix} 1 \\ -1 \\ \vdots \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -4 \\ \vdots \\ 4 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ n \\ -n \\ -(n-1) \\ \vdots \\ -2 \\ -1 \end{bmatrix}, \text{ or } \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

□

8. Prove

$$\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 = 2\|\mathbf{v}\|^2 + 2\|\mathbf{w}\|^2.$$

Solution.

Suppose $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^n$. We have

$$\begin{aligned}
\|\mathbf{v}+\mathbf{w}\|^2 + \|\mathbf{v}-\mathbf{w}\|^2 &= (\mathbf{v}+\mathbf{w}) \cdot (\mathbf{v}+\mathbf{w}) + (\mathbf{v}-\mathbf{w}) \cdot (\mathbf{v}-\mathbf{w}) \\
&= \sum_{i=1}^{n} (v_i + w_i)^2 + \sum_{i=1}^{n} (v_i - w_i)^2 \\
&= \sum_{i=1}^{n} v_i^2 + 2v_i w_i + w_i^2 + \sum_{i=1}^{n} v_i^2 - 2v_i w_i + w_i^2 \\
&= \sum_{i=1}^{n} 2v_i^2 + 2w_i^2 \\
&= 2\|\mathbf{v}\|^2 + 2\|\mathbf{w}\|^2.
\end{aligned}$$

□

9. Prove
$$\|\mathbf{v}-\mathbf{w}\|^2 = \|\mathbf{v}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\| \cos\theta + \|\mathbf{w}\|^2.$$

Solution.

Suppose $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^n$. We have

$$\begin{aligned}
\|\mathbf{v}-\mathbf{w}\|^2 &= (\mathbf{v}-\mathbf{w}) \cdot (\mathbf{v}-\mathbf{w}) \\
&= \sum_{i=1}^{n} (v_i - w_i)^2 \\
&= \sum_{i=1}^{n} v_i^2 - 2v_i w_i + w_i^2 \\
&= \|\mathbf{v}\|^2 - 2\sum_{i=1}^{n} v_i w_i + \|\mathbf{w}\|^2 \\
&= \|\mathbf{v}\|^2 - 2\mathbf{v} \cdot \mathbf{w} + \|\mathbf{w}\|^2 \\
&= \|\mathbf{v}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\| \cos\theta + \|\mathbf{w}\|^2.
\end{aligned}$$

□

10. Let $\mathbf{w} = (a, b, c)$ and $\mathbf{v} = (c, b, a)$ where $a + b + c = 0$. Find the angle between $\mathbf{w}$ and $\mathbf{v}$.

Solution.

We know that
$$\cos\theta = \frac{\mathbf{w} \cdot \mathbf{v}}{\|\mathbf{w}\|\|\mathbf{v}\|}$$

where θ is the angle between $\mathbf{w}$ and $\mathbf{v}$. Now

$$\|w\|\|v\| = \sqrt{a^2+b^2+c^2}\sqrt{c^2+b^2+a^2} = a^2+b^2+c^2$$

and

$$\mathbf{w}\cdot\mathbf{v} = 2ac+b^2.$$

Then, we have that

$$\theta = \cos^{-1}\left(\frac{2ac+b^2}{a^2+b^2+c^2}\right).$$

Note: This problem had a typo. It was inteded to have $\mathbf{v} = (c, a, b)$. Then, we find that

$$\mathbf{w}\cdot\mathbf{v} = ac+ab+cb.$$

Since $a+b+c=0$, we find that $a^2+b^2+c^2 = -2ab-2ac-2bc$. Therefore we find that $\cos\theta = -\frac{1}{2}$. □