MAT 22A Problem Set 2 (Due 6/29 8 AM)

1. Let
$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$
Find $\mathbf{x}$ so that
$$A\mathbf{x} = \mathbf{b}.$$
What is the matrix A^{-1}? Are the columns of A independent or dependent? Why?

2. The sum matrix $A \in \mathbb{R}^{n\times n}$ is the matrix with entries
$$a_{ij} = \begin{cases} 1 & i \geq j \\ 0 & \text{otherwise.} \end{cases}$$
Write down the form of A. Let $B \in \mathbb{R}^{n\times n}$ be the difference matrix
$$b_{ij} = \begin{cases} 1 & i = j \\ -1 & i = j+1 \\ 0 & \text{otherwise.} \end{cases}$$
Show that if $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x} = B\mathbf{b}$. That is, B is the inverse of A.

3. Let $f(t)$ be a twice differentiable function. The centered difference for $f''(t)$ is
$$f''(t) \approx \frac{f(t+h) - 2f(t) + f(t-h)}{h^2}.$$
Let $t_i = ih$ for $i = 1, 2, \dots n$. Find the centered difference matrix $A \in \mathbb{R}^{n\times n}$ so that $\mathbf{f}'' = A\mathbf{f} + \frac{1}{h^2}\mathbf{b}$ where
$$\mathbf{f}'' = \begin{bmatrix} f''(t_2) \\ f''(t_3) \\ \vdots \\ f''(t_{n-1}) \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} f(t_2) \\ f(t_3) \\ \vdots \\ f(t_{n-1}) \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} f(t_1) \\ 0 \\ \vdots \\ 0 \\ f(t_n) \end{bmatrix}.$$
What are the entries of A (that is, what are the a_{ij}), and what is the form of A?

4. Consider the cyclic differences matrix $C \in \mathbb{R}^{n\times n}$ where
$$c_{ij} = \begin{cases} 1 & i = j \\ -1 & i-1 = j, \quad \text{for } i = 2, \dots, n \\ -1 & i = 1, j = n \\ 0 & \text{otherwise.} \end{cases}$$
Write down the structure of C. Find $\mathbf{x}$ so that $C\mathbf{x} = \mathbf{0}$. How many solutions are there?

5. For what values of c give dependent columns so that a linear combinations of columns produces the zero vector. For which values of c are they independent?

   (a) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & c \end{bmatrix}$

   (b) $\begin{bmatrix} c & c & c \\ 2 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

6. Compute $A\mathbf{x}$ using the dot product perspective and the linear combination perspective for the following:

   (a)
$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

   (b)
$$\begin{bmatrix} -1 & 0 & 2 & 5 \\ 2 & -1 & 2 & 2 \\ 1 & 0 & -2 & 4 \\ 3 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 4 \end{bmatrix}$$

7. Write down the matrix form and draw the row picture and column picture for the following systems:

   (a)
$$\begin{aligned} x + y &= 2 \\ x - y &= 1 \end{aligned}$$

   (b)
$$\begin{aligned} 2x + y &= 2 \\ x - 2y &= 4 \end{aligned}$$

8. Let A_i be the matrix that projects $\mathbf{x} \in \mathbb{R}^n$ onto the $n-1$ dimensional space so that $x_i = 0$. What is the form of A_i? What happens when you consider the following iteration: Let $\mathbf{b}_1 = A_1\mathbf{x}$. For $i = 2, 3, \ldots, n$: $\mathbf{b}_i = A_i\mathbf{b}_{i-1}$. What is $\mathbf{b}_n$?

9. Find a matrix $P \in \mathbb{R}^{3\times 3}$ so that

$$P\mathbf{x} = \begin{bmatrix} y \\ z \\ x \end{bmatrix}$$

where $\mathbf{x} = (x, y, z)$. Find $Q \in \mathbb{R}^{3\times 3}$ so that

$$Q \begin{bmatrix} y \\ z \\ x \end{bmatrix} = \mathbf{x}.$$

10. A magic square is an $n \times n$ array of numbers such that the row, column, and diagonals sum to the same number and all the numbers $1, 2, \ldots, n^2$ are used. Let $M \in \mathbb{R}^{n\times n}$ be a magic square. Find a magic square for $n = 5$. What is $M\mathbf{x}$ when $\mathbf{x} = (1, 1, 1, 1, 1)$. What is $M\mathbf{x}$ when $\mathbf{x} = (1, 1, \ldots, 1)$ for general n?