MAT 22A Problem Set 2 Solutions

1. Let
$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Find $\mathbf{x}$ so that
$$A\mathbf{x} = \mathbf{b}.$$

What is the matrix A^{-1}? Are the columns of A independent or dependent? Why?

Solution.

We have the system of equations
$$\begin{aligned} 2x_1 + 2x_2 + x_3 &= b_1 \\ -x_2 &= b_2 \\ x_1 - x_3 &= b_3. \end{aligned}$$

We immediately see that $x_2 = -b_2$. Next, $x_3 = x_1 - b_3$. Now, from the first equation, we have
$$\begin{aligned} x_1 + 2x_2 + x_3 &= b_1 \\ x_1 - 2b_2 + x_1 - b_3 &= b_1 \\ 2x_1 &= b_1 + 2b_2 + b_3 \\ x_1 &= \frac{1}{2}(b_1 + 2b_2 + b_3) \end{aligned}$$

and so $x_3 = \frac{1}{2}b_1 + b_2 - \frac{1}{2}b_3$. Therefore, we find that
$$\mathbf{x} = \begin{bmatrix} \frac{1}{2}b_1 + b_2 + \frac{1}{2}b_3 \\ -b_2 \\ \frac{1}{2}b_1 + b_2 - \frac{1}{2}b_3 \end{bmatrix}.$$

Now, $\mathbf{x} = A^{-1}\mathbf{b}$,so
$$A^{-1}\mathbf{b} = \begin{bmatrix} \frac{1}{2}b_1 + b_2 + \frac{1}{2}b_3 \\ -b_2 \\ \frac{1}{2}b_1 + b_2 - \frac{1}{2}b_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Thus,
$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}.$$

Since we were able to find A^{-1}, the columns of A are linearly independent.

□

2. The sum matrix $A \in \mathbb{R}^{n\times n}$ is the matrix with entries

$$a_{ij} = \begin{cases} 1 & i \geq j \\ 0 & \text{otherwise.} \end{cases}$$

Write down the form of A. Let $B \in \mathbb{R}^{n\times n}$ be the difference matrix

$$b_{ij} = \begin{cases} 1 & i = j \\ -1 & i = j+1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that if $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x} = B\mathbf{b}$. That is, B is the inverse of A.

Solution.

The matrix A is of the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. We see that

$$A\mathbf{x} = \begin{bmatrix} x_1 \\ x_1 + x_2 \\ \vdots \\ \sum_{k=1}^{i} x_k \\ \vdots \\ \sum_{k=1}^{n} x_k \end{bmatrix}$$

so

$$A\mathbf{x} = \begin{bmatrix} x_1 \\ x_1 + x_2 \\ \vdots \\ \sum_{k=1}^{i} x_k \\ \vdots \\ \sum_{k=1}^{n} x_k \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}.$$

Now, solving from the first row and working down, we see that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - b_1 \\ b_3 - b_2 \\ \vdots \\ b_i - b_{i-1} \\ \vdots \\ b_n - b_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -1 & 1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}.$$

Thus, we see that

$$\mathbf{x} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -1 & 1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \mathbf{b} = A^{-1}\mathbf{b}.$$

Comparing the matrix that we found to the difference matrix, we see that matrix we found is indeed the difference matrix. Therefore, $A^{-1} = B$. $\square$

3. Let $f(t)$ be a twice differentiable function. The centered difference for $f''(t)$ is

$$f''(t) \approx \frac{f(t+h) - 2f(t) + f(t-h)}{h^2}.$$

Let $t_i = ih$ for $i = 1, 2, \ldots n$. Find the centered difference matrix $A \in \mathbb{R}^{n \times n}$ so that $\mathbf{f}'' = A\mathbf{f} + \frac{1}{h^2}\mathbf{b}$ where

$$\mathbf{f}'' = \begin{bmatrix} f''(t_2) \\ f''(t_3) \\ \vdots \\ f''(t_{n-1}) \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} f(t_2) \\ f(t_3) \\ \vdots \\ f(t_{n-1}) \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} f(t_1) \\ 0 \\ \vdots \\ 0 \\ f(t_n) \end{bmatrix}.$$

What are the entries of A (that is, what are the a_{ij}), and what is the form of A?

Solution.

First notice that $t_i + h = ih + h = (i+1)h = t_{i+1}$ and similarly, we have that $t_i - h = t_{i-1}$. Now,

$$f''(t_i) = \frac{f(t_i + h) - 2f(t_i) + f(t_i - h)}{h^2} = \frac{f(t_{i+1}) - 2f(t_i) + f(t_{i-1})}{h^2}.$$

Now, writing out a few rows, we have

$$\mathbf{f}'' = \begin{bmatrix} \frac{f(t_3) - 2f(t_2) + f(t_1)}{h^2} \\ \frac{f(t_4) - 2f(t_3) + f(t_2)}{h^2} \\ \vdots \\ \frac{f(t_n) - 2f(t_{n-1}) + f(t_{n-2})}{h^2} \end{bmatrix}.$$

We want to find a matrix A so that $\mathbf{f}'' = A\mathbf{f} + \frac{1}{h^2}\mathbf{b}$. Since $f(t_1)$ and $f(t_n)$ do not appear in the vector $\mathbf{f}$, let us rewrite $\mathbf{f}''$ as

$$\mathbf{f}'' = \begin{bmatrix} \frac{f(t_3)-2f(t_2)+f(t_1)}{h^2} \\ \frac{f(t_4)-2f(t_3)+f(t_2)}{h^2} \\ \vdots \\ \frac{f(t_n)-2f(t_{n-1})+f(t_{n-2})}{h^2} \end{bmatrix} = \begin{bmatrix} \frac{f(t_3)-2f(t_2)}{h^2} \\ \frac{f(t_4)-2f(t_3)+f(t_2)}{h^2} \\ \vdots \\ \frac{-2f(t_{n-1})+f(t_{n-2})}{h^2} \end{bmatrix} + \begin{bmatrix} \frac{f(t_1)}{h^2} \\ 0 \\ \vdots \\ 0 \\ \frac{f(t_n)}{h^2} \end{bmatrix}.$$

Now, we want find the matrix A. Thinking about the dot product perspective of vector-matrix multiplication, we find that

$$\mathbf{f}'' = \begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} & 0 & \cdots & 0 \\ \frac{1}{h^2} & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{1}{h^2} \\ 0 & \cdots & 0 & \frac{1}{h^2} & -\frac{2}{h^2} \end{bmatrix} \mathbf{f} + \begin{bmatrix} \frac{f(t_1)}{h^2} \\ 0 \\ \vdots \\ 0 \\ \frac{f(t_n)}{h^2} \end{bmatrix}.$$

The entries of this matrix are

$$a_{ij} = \begin{cases} -\frac{2}{h^2} & i = j \\ \frac{1}{h^2} & i-1 = j \\ \frac{1}{h^2} & i+1 = j \\ 0 & \text{otherwise.} \end{cases}$$

□

4. Consider the cyclic differences matrix $C \in \mathbb{R}^{n \times n}$ where

$$c_{ij} = \begin{cases} 1 & i = j \\ -1 & i-1 = j, \quad \text{for } i = 2, \ldots, n \\ -1 & i = 1, j = n \\ 0 & \text{otherwise.} \end{cases}$$

Write down the structure of C. Find $\mathbf{x}$ so that $C\mathbf{x} = \mathbf{0}$. How many solutions are there?

Solution.

The structure of C is

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 & -1 \\ -1 & 1 & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}.$$

Now, if we consider $C\mathbf{x} = \mathbf{0}$, first we see that

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & -1 \\ -1 & 1 & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 - x_n \\ x_2 - x_1 \\ \vdots \\ x_i - x_{i-1} \\ \vdots \\ x_n - x_{n-1} \end{bmatrix}.$$

Now,

$$\begin{bmatrix} x_1 - x_n \\ x_2 - x_1 \\ \vdots \\ x_i - x_{i-1} \\ \vdots \\ x_n - x_{n-1} \end{bmatrix} = \mathbf{0}$$

so we find that

$$\begin{aligned} x_1 &= x_n \\ x_2 &= x_1 \\ &\vdots \\ x_i &= x_{i-1} \\ &\vdots \\ x_n &= x_{n-1}. \end{aligned}$$

This means that $x_1 = x_2 = \cdots = x_n$. Therefore any vector of the form

$$c \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

where $c \in \mathbb{R}$ would be a solution, so there are infinitely many solutions! □

5. For what values of c give dependent columns so that a linear combinations of columns produces the zero vector. For which values of c are they independent?

   (a) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & c \end{bmatrix}$

(b) $\begin{bmatrix} c & c & c \\ 2 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Solution.

(a) We have the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & c \end{bmatrix}.$$

The columns are dependent if we can find $\mathbf{x} \neq \mathbf{0}$ so that

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & c \end{bmatrix} \mathbf{x} = \mathbf{0}.$$

We have the system of equations

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 + 2x_2 + x_3 &= 0 \\ -2x_1 + x_2 + cx_3 &= 0. \end{aligned}$$

The first equation tells us $x_1 = -x_2$, and the second equation then tells us that $x_3 = -x_1 - 2x_2 = -x_2$. Now, using the third equation, we have

$$\begin{aligned} -2x_1 + x_2 + cx_3 &= 0 \\ 2x_2 + x_2 - cx_2 &= 0 \\ (3 - c)x_2 &= 0. \end{aligned}$$

We see that if $c \neq 3$, then we must have $x_2 = 0$. Then, $x_3 = x_1 = 0$ and we have that $\mathbf{x} = 0$. If $c \neq 3$, then we see that the columns are independent. If $c = 3$, then the columns are dependent.

(b) We again want to consider the equation

$$\begin{bmatrix} c & c & c \\ 2 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}.$$

We have the system of equations

$$\begin{aligned} c(x_1 + x_2 + x_3) &= 0 \\ 2x_1 + x_2 - x_3 &= 0 \\ -x_1 + x_2 &= 0. \end{aligned}$$

From the third equation, we see that $x_1 = x_2$, and the second equation tells us that $x_3 = 2x_1 + x_2 = 3x_1$. Now, using the first equation, we have

$$\begin{aligned} c(x_1 + x_2 + x_3) &= 0 \\ c(x_1 + x_1 + 3x_1) &= 0 \\ 5cx_1 &= 0. \end{aligned}$$

We see that if $c \neq 0$, then we must have that $x_1 = 0$ so $x_2 = x_3 = 0$. Then, the columns of A are dependent when $c = 0$ and independent when $c \neq 0$.

(c) Again, we consider the equations

$$\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

which leads to the system of equations

$$\begin{aligned} x_1 + cx_3 &= 0 \\ x_2 &= 0 \\ -x_1 + x_3 &= 0. \end{aligned}$$

We see that $x_0 = 0$ and the third equation tells us that $x_3 = x_1$. Now, from the first equaiton, we have

$$\begin{aligned} x_1 + cx_3 &= 0 \\ x_3 + cx_3 &= 0 \\ (1 + c)x_3 &= 0. \end{aligned}$$

If $c \neq -1$, then we must have that $x_3 = 0$. Then, $x_1 = 0$ and we see that $\mathbf{x} = \mathbf{0}$. Therefore the columns are dependent when $c = -1$ and independent when $c \neq -1$.

□

6. Compute $A\mathbf{x}$ using the dot product perspective and the linear combination perspective for the following:

(a)

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

(b)

$$\begin{bmatrix} -1 & 0 & 2 & 5 \\ 2 & -1 & 2 & 2 \\ 1 & 0 & -2 & 4 \\ 3 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 4 \end{bmatrix}$$

Solution.

(a) Dot product perspective:

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} (1,0,2,1) \cdot (1,1,2,3) \\ (1,0,0,0) \cdot (1,1,2,3) \\ (-1,1,1,0) \cdot (1,1,2,3) \\ (1,0,3,2) \cdot (1,1,2,3) \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 2 \\ 13 \end{bmatrix}$$

Linear combination perspective:

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 2 \\ 6 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 2 \\ 13 \end{bmatrix}$$

(b) Dot product perspective:

$$\begin{bmatrix} -1 & 0 & 2 & 5 \\ 2 & -1 & 2 & 2 \\ 1 & 0 & -2 & 4 \\ 3 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} (-1,0,2,5) \cdot (-1,1,0,4) \\ (2,-1,2,2) \cdot (-1,1,0,4) \\ (1,0,-2,4) \cdot (-1,1,04) \\ (3,2,2,1) \cdot (-1,1,0,4) \end{bmatrix} = \begin{bmatrix} 21 \\ 5 \\ 15 \\ 3 \end{bmatrix}$$

Linear combination perspective:

$$\begin{aligned}
\begin{bmatrix} -1 & 0 & 2 & 5 \\ 2 & -1 & 2 & 2 \\ 1 & 0 & -2 & 4 \\ 3 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 4 \end{bmatrix} &= -1 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 2 \\ -2 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 5 \\ 2 \\ 4 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ -2 \\ -1 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 20 \\ 8 \\ 16 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} 21 \\ 5 \\ 15 \\ 3 \end{bmatrix}
\end{aligned}$$

□

7. Write down the matrix form and draw the row picture and column picture for the following systems:

(a)

$$\begin{aligned} x + y &= 2 \\ x - y &= 1 \end{aligned}$$

(b)

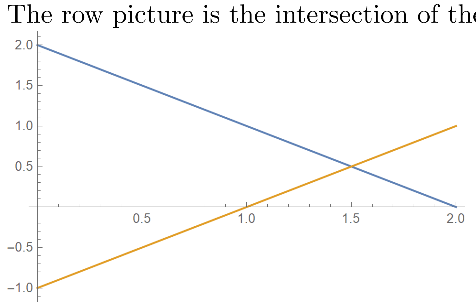
$$\begin{aligned} 2x + y &= 2 \\ x - 2y &= 4 \end{aligned}$$

Solution.

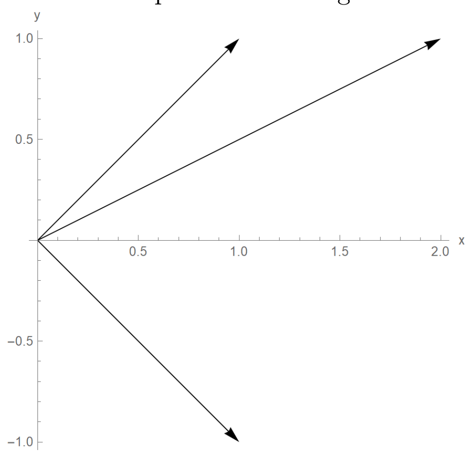
(a) Matrix form:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

The row picture is the intersection of the lines:



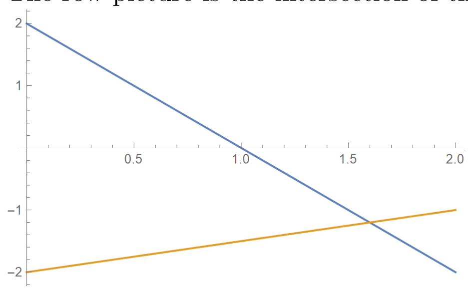
The column picture is finding a linear combination of vectors:



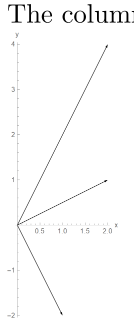
(b) Matrix form:

$$\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

The row picture is the intersection of the lines:



The column picture is finding a linear combination of vectors:



□

8. Let A_i be the matrix that projects $\mathbf{x} \in \mathbb{R}^n$ onto the $n-1$ dimensional space so that $x_i = 0$. What is the form of A_i? What happens when you consider the following iteration: Let $\mathbf{b}_1 = A_1\mathbf{x}$. For $i = 2, 3, \ldots, n$: $\mathbf{b}_i = A_i\mathbf{b}_{i-1}$. What is $\mathbf{b}_n$?

Solution.

Since

$$A_i\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{i-1} \\ 0 \\ x_{i+1} \\ \vdots \\ x_n \end{bmatrix}.$$

We see that A_i makes the i^{th} component of $\mathbf{x}$ zero. Thinking about the dot product of $\mathbf{x}$ with the i^{th} row of A_i, we can get the desired result by making all the entries in the i^{th} row of A_i zero. For the other rows, we know that $I\mathbf{x} = \mathbf{x}$, so we see that

$$A_i = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 & 0\cdots & 0 \\ \vdots & & & & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

Now, we consider the iteration scheme. We see that $A_1\mathbf{x}$ makes $x_1 = 0$ in the resulting vector. Then $A_2A_1\mathbf{x}$ makes $x_1 = x_2 = 0$ in the resulting vector. Continuing the iteration, you find that $\mathbf{b}_n = \mathbf{0}$.

□

9. Find a matrix $P \in \mathbb{R}^{3\times 3}$ so that

$$P\mathbf{x} = \begin{bmatrix} y \\ z \\ x \end{bmatrix}$$

where $\mathbf{x} = (x, y, z)$. Find $Q \in \mathbb{R}^{3\times 3}$ so that

$$Q\begin{bmatrix} y \\ z \\ x \end{bmatrix} = \mathbf{x}.$$

Solution.

We want to find P so that

$$P \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix}.$$

Thinking about the dot product perspective of vector-matrix multiplication, we see that

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Notice that P is the permutation matrix! Now, we want to find Q. Thinking about which elements we want to select in each row, we see that

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

□

10. A magic square is an $n \times n$ array of numbers such that the row, column, and diagonals sum to the same number and all the numbers $1, 2, \ldots, n^2$ are used. Let $M \in \mathbb{R}^{n \times n}$ be a magic square. Find a magic square for $n = 5$. What is $M\mathbf{x}$ when $\mathbf{x} = (1, 1, 1, 1, 1)$. What is $M\mathbf{x}$ when $\mathbf{x} = (1, 1, \ldots, 1)$ for general n?

    Solution.

    A possible magic square can be found at `https://en.wikipedia.org/wiki/Magic_square`. $M\mathbf{x}$ gives you the row sum of the matrix M. This is also the column and diagonal sum. The same holds true for general M since

$$\begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ \vdots & \vdots & & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} m_{1j} \\ \vdots \\ \sum_{j=1}^{n} m_{ij} \\ \vdots \\ \sum_{j=1}^{n} m_{nj} \end{bmatrix}.$$

□