MAT 22A Problem Set 3 Solutions

1. Find the multipliers ℓ_{ij} to reduce the following systems to an upper triangular system:

   (a)

$$\begin{aligned} 2x + 3y + z &= 1 \\ 2x + 3y - z &= 2 \\ 3x + y + 2z &= 1 \end{aligned}$$

   (b)

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

Solution.

(a) First, we want to clear the first column. We see that $\ell_{21} = 1$ and $\ell_{31} = \frac{3}{2}$. We get the new system

$$\begin{aligned} 2x + 3y + z &= 1 \\ -2z &= 1 \\ -\frac{7}{2}y + \frac{1}{2}z &= -\frac{1}{2}. \end{aligned}$$

We see that our next potential pivot is 0, so we swap the second and third equation to get

$$\begin{aligned} 2x + 3y + z &= 1 \\ -\frac{7}{2}y + \frac{1}{2}z &= -\frac{1}{2} \\ -2z &= 1. \end{aligned}$$

We see that we have arrived at an upper triangular system, so the multiplier that were needed were $\ell_{21} = 1$ and $\ell_{31} = \frac{3}{2}$.

(b) To clear the first column, we see that $\ell_{21} = \frac{d}{a}$ Then, we get the new system

$$\begin{aligned} ax + by &= c \\ (e - \frac{bd}{a})y &= f - \frac{cd}{a}. \end{aligned}$$

We see that we have arrived at an upper triangular system, so the only multiplier that we needed was $\ell_{21} = \frac{d}{a}$.

□

2. Solve the following system using elimination to reduce the system to a triangular system. Then use back substitution to find the solution.

$$\begin{aligned} x + y + z &= -1 \\ 2x - y - 3z - t &= 1 \\ x + y + t &= 2 \\ 4x - z + 2t &= -1 \end{aligned}$$

Solution.

Our first pivot is 1, and the multipliers are $\ell_{21} = 2$, $\ell_{31} = 1$, and $\ell_{41} = 4$. Performing the elimination, we get the system

$$\begin{aligned} x + y + z &= -1 \\ -3y - 5z - t &= 3 \\ -z + t &= 3 \\ -4y - 5z + 2t &= 3. \end{aligned}$$

Out next pivot is -3, and the multiplier is $\ell_{42} = \frac{4}{3}$. Performing the elimination, we get the system

$$\begin{aligned} x + y + z &= -1 \\ -3y - 5z - t &= 3 \\ -z + t &= 3 \\ \frac{5}{3}z + \frac{10}{3}t &= -1. \end{aligned}$$

The next pivot is -1, and the multiplier is $-\frac{5}{3}$. Performing the elimination, we get the system

$$\begin{aligned} x + y + z &= -1 \\ -3y - 5z - t &= 3 \\ -z + t &= 3 \\ 5t &= 4. \end{aligned}$$

Now, using back substitution, we find

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} -\frac{6}{5} \\ \frac{12}{5} \\ -\frac{11}{5} \\ \frac{4}{5} \end{bmatrix}.$$

□

3. Find c so that the linear system below

(a) requires a row exchange.
(b) is singular.
(c) does not require a row exchange.

$$\begin{aligned} x + 2y + z &= 4 \\ 3x + cy + 3z &= 2 \\ -y + z &= 1 \end{aligned}$$

Finally, find the solution to the system in terms of c.

Solution.

(a) We will need to perform a row exchange if we find that one of our pivots are zero. Our first pivot is 1 and so we that our first multiplier is $\ell_{21} = 3$. Performing elimination, we get the system

$$\begin{aligned} x + 2y + z &= 4 \\ (c-6)y &= -10 \\ -y + z &= 1. \end{aligned}$$

We see that if $c = 6$, we require a row exchange. Note that we see that the system is inconsistent when $c = 6$.

(b) After the first step of elimination and performing a row exchange, we get the system

$$\begin{aligned} x + 2y + z &= 4 \\ -y + z &= 1 \\ (c-6)y &= -10. \end{aligned}$$

Now, our pivot is -1 and the multiplier is $\ell_{32} = -(c-6)$. Performing elimination, we get the system

$$\begin{aligned} x + 2y + z &= 4 \\ -y + z &= 1 \\ (c-6)z &= -16 + c. \end{aligned}$$

We find that $z = \frac{-16+c}{c-6}$. We see that the system has no solution if $c = 6$. Note that when $c = 6$, we get an equation like $0 = -10$, and so this indicates that there are no solutions.

(c) From (a), we see that no row exchanges are required if $c \neq 6$.

(d) Solve the system using back subsitution on the upper triangular system from part (b), we find that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 - \frac{2(-16+c)}{c-6} - \frac{-16+c}{c-6} \\ \frac{-16+c}{c-6} - 1 \\ \frac{-16+c}{c-6} \end{bmatrix}.$$

□

4. Let $A \in \mathbb{R}^{n\times n}$. Use the matrix-matrix multiplication definition to show that $AI = A$ where I is the identity matrix.

   Solution.

   We know that the identity matrix has entries that are 1 if $i = j$ and 0 otherwise. Let B = AI, then

$$b_{ij} = \sum_{k=1}^{n} a_{ik}(I)_{kj}.$$

   Every term in the sum is 0 except when $k = j$ and so we have that

$$\begin{aligned} b_{ij} &= \sum_{k=1}^{n} a_{ik}(I)_{kj} \\ &= a_{ij}(I)_{jj} \\ &= a_{ij}. \end{aligned}$$

   We see that $b_{ij} = a_{ij}$ for all i, j so $B = A$. □

5. Let $A \in \mathbb{R}^{m\times n}$ and $B, C \in \mathbb{R}^{n\times \ell}$. Use the matrix-matrix multiplication definition to show that $A(B + C) = AB + AC$.

   Solution.

   Let $D = A(B + C)$. Then

$$\begin{aligned} d_{ij} &= \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj}) \\ &= \sum_{k=1}^{n} a_{ik}b_{kj} + a_{ik}c_{kj} \\ &= \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}. \end{aligned}$$

   We see that the sums on the LHS are the products AB and AC, so we see that indeed $A(B + C) = AB + AC$. □

6. Find an example of A and B (4×4 matrices) such that

   (a) $AB = BA$.

   (b) $AB \neq BA$.

   Assume A and B are not the zero matrix or the identity matrix.

Solution.

(a) $AB = BA$.

$$\begin{bmatrix}1&0&0&0\\0&2&0&0\\0&0&3&0\\0&0&0&4\end{bmatrix}\begin{bmatrix}2&0&0&0\\0&3&0&0\\0&0&4&0\\0&0&0&5\end{bmatrix}=\begin{bmatrix}2&0&0&0\\0&6&0&0\\0&0&12&0\\0&0&0&20\end{bmatrix}$$

and

$$\begin{bmatrix}2&0&0&0\\0&3&0&0\\0&0&4&0\\0&0&0&5\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&2&0&0\\0&0&3&0\\0&0&0&4\end{bmatrix}=\begin{bmatrix}2&0&0&0\\0&6&0&0\\0&0&12&0\\0&0&0&20\end{bmatrix}.$$

(b) $AB \neq BA$.

$$\begin{bmatrix}2&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&2&3&4\\1&0&1&1\\0&1&1&1\\1&0&0&1\end{bmatrix}=\begin{bmatrix}2&4&6&8\\1&0&1&1\\0&1&1&1\\1&0&0&1\end{bmatrix}$$

but

$$\begin{bmatrix}1&2&3&4\\1&0&1&1\\0&1&1&1\\1&0&0&1\end{bmatrix}\begin{bmatrix}2&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}2&2&3&4\\2&0&1&1\\0&1&1&1\\2&0&0&1\end{bmatrix}$$

□

7. Consider the following system of linear equations

$$\begin{aligned}2y - z + t &= 1\\ 3x + 4y - z &= 1\\ x - 2y + t &= 1\\ x + t &= 1.\end{aligned}$$

Solve the system using a squence of elimination matrices E_{ij} and permutation matrices P_{ij}. Let C be the product of the matrices used to reduce the system to triangular form. Compute C and show that CA is an upper triangular matrix where A is the coefficient matrix.

Solution.

We have the matrix system

$$\begin{bmatrix}0&2&-1&1\\3&4&-1&0\\1&-2&0&1\\1&0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\\t\end{bmatrix}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}.$$

We see that our potential pivot is 0, so we permute the first and second rows using a permutation matrix P_{12} and so

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & -1 & 1 \\ 3 & 4 & -1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & -1 & 0 \\ 0 & 2 & -1 & 1 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

We see that our first pivot is 3, and our multipliers are $\ell_{31} = \frac{1}{3}$ and $\ell_{41} = \frac{1}{3}$. Now,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{3} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & -1 & 0 \\ 0 & 2 & -1 & 1 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & -1 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & -\frac{10}{3} & \frac{1}{3} & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{3} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & -1 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & -\frac{10}{3} & \frac{1}{3} & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & -1 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & -\frac{10}{3} & \frac{1}{3} & 1 \\ 0 & -\frac{4}{3} & \frac{1}{3} & 1 \end{bmatrix}.$$

Our next pivot is 2. The multipliers are $\ell_{32} = -\frac{5}{3}$ and $\ell_{42} = -\frac{2}{3}$. Now,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{5}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & -1 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & -\frac{10}{3} & \frac{1}{3} & 1 \\ 0 & -\frac{4}{3} & \frac{1}{3} & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & -1 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -\frac{4}{3} & \frac{8}{3} \\ 0 & -\frac{4}{3} & \frac{1}{3} & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{2}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & -1 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -\frac{4}{3} & \frac{8}{3} \\ 0 & -\frac{4}{3} & \frac{1}{3} & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & -1 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -\frac{4}{3} & \frac{8}{3} \\ 0 & 0 & -\frac{1}{3} & \frac{5}{3} \end{bmatrix}.$$

The next pivot is $-\frac{4}{3}$ and the multiplier $\ell_{43} = \frac{1}{4}$. Now,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & -1 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -\frac{4}{3} & \frac{8}{3} \\ 0 & 0 & -\frac{1}{3} & \frac{5}{3} \end{bmatrix} = \begin{bmatrix} 3 & 4 & -1 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -\frac{4}{3} & \frac{8}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We have arrived at our upper triangular system. Now, the matrix C is

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{2}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{5}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{3} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{3} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{5}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{3} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{3} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{5}{3} & 1 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{3} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{3} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{5}{3} & 1 & 0 \\ -\frac{1}{3} & \frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{3} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{3} & \frac{5}{3} & 1 & 0 \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{5}{3} & -\frac{1}{3} & 1 & 0 \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix}$$

Finally,

$$CA = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{5}{3} & -\frac{1}{3} & 1 & 0 \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & -1 & 1 \\ 3 & 4 & -1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & -1 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -\frac{4}{3} & \frac{8}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

□

8. Let

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}.$$

Compute A^n.

Solution.

We see that

$$A^2 = \begin{bmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2a & a^2+2b \\ 0 & 1 & 2a \\ 0 & 0 & 1 \end{bmatrix},$$

$$A^3 = \begin{bmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2a & a^2+2b \\ 0 & 1 & 2a \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3a & 3a^2+3b \\ 0 & 1 & 3a \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$A^4 = \begin{bmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3a & 3a^2+3b \\ 0 & 1 & 3a \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4a & 6a^2+4b \\ 0 & 1 & 4a \\ 0 & 0 & 1 \end{bmatrix}.$$

We notice that the upper corner of A^n accumulates new terms, and we see that

$$A^n = \begin{bmatrix} 1 & na & \left(\frac{(n-2)(n-1)}{2} + (n-1)\right) a^2 + nb \\ 0 & 1 & na \\ 0 & 0 & 1 \end{bmatrix}.$$

□

9. Let $A \in \mathbb{R}^{n \times n}$ with entries

$$a_{ij} = \begin{cases} 1 & i = j-1 \\ 0 & \text{otherwise.} \end{cases}$$

Compute

(a) A^2,

(b) A^{n-1},

(c) A^n.

Solution.

First note that A has the form

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}.$$

(a) To compute A^2, examine the 4×4 case, and examining the $n \times n$ case, we see that

$$A^2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & 1 \\ \vdots & & & & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}.$$

(b) To compute A^{n-1}, again examine the 4×4 case, then we see that for the $n \times n$ case,

$$A^{n-1} = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & \ddots & & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}.$$

(c) To compute A^n, examine the 4×4 case, and examining the $n \times n$ case, we see that

$$A^{n} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}.$$

□

10. Compute AB using the 4 different perspectives of matrix-matrix multiplication

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution.

(a) Dot product perspective:

$$
\begin{aligned}
AB &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} (1,-1,0)\cdot(2,1,0) & (1,-1,0)\cdot(1,0,0) & (1,-1,0)\cdot(-1,1,1) \\ (0,1,-1)\cdot(2,1,0) & (0,1,-1)\cdot(1,0,0) & (0,1,-1)\cdot(-1,1,1) \\ (1,1,1)\cdot(2,1,0) & (1,1,1)\cdot(1,0,0) & (1,1,1)\cdot(-1,1,1) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix}.
\end{aligned}
$$

(b) A times columns of B perspective:

$$
\begin{aligned}
AB &= \begin{bmatrix} A\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} & A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & A\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

(c) Rows of A times B perspective:

$$
\begin{aligned}
AB &= \begin{bmatrix} \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} B \\ \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} B \\ \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} B \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix}.
\end{aligned}
$$

(d) Columns of A times rows of B perspective:

$$
\begin{aligned}
AB &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 0 \\ 2 & 1 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix}.
\end{aligned}
$$

□