MAT 22A Problem Set 4 Solutions

1. Compute the inverse of the following matrices using Gauss-Jordan elimination and the augmented matrix:

   (a) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 2 & -1 \\ 1 & 4 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

   Solution.

   (a)

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] \begin{array}{l} \\ r_2 - \frac{3}{2}r_1 \end{array} = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right] \begin{array}{l} r_1 - 2r_2 \\ \\ \end{array}$$

$$= \left[\begin{array}{cc|cc} 2 & 0 & 4 & -2 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right] \begin{array}{l} \frac{1}{2}r_1 \\ 2r_2 \end{array}$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -3 & 2 \end{array}\right]$$

   so

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$$

   (b)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ r_2 - 2r_1 \end{array} = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} r_1 + r_3 \\ r_2 + r_3 \\ \\ \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ -r_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right]$$

   so

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix}.$$

(c)

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ r_2 - r_1 \\ r_3 + r_1 \end{array} = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} r_1 - r_2 \\ \\ r_3 - \frac{3}{2}r_2 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 2 & -1 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 \\ 0 & 0 & -2 & \frac{5}{2} & -\frac{3}{2} & 1 \end{array}\right] \begin{array}{l} r_1 - \frac{3}{2}r_3 \\ r_2 + r_3 \\ \\ \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{7}{4} & \frac{5}{4} & -\frac{3}{2} \\ 0 & 2 & 0 & \frac{3}{2} & -\frac{1}{2} & 1 \\ 0 & 0 & -2 & \frac{5}{2} & -\frac{3}{2} & 1 \end{array}\right] \begin{array}{l} \\ \frac{1}{2}r_2 \\ -\frac{1}{2}r_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{7}{4} & \frac{5}{4} & -\frac{3}{2} \\ 0 & 1 & 0 & \frac{3}{4} & -\frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{5}{4} & \frac{3}{4} & -\frac{1}{2} \end{array}\right]$$

so

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 4 & 1 \\ -1 & 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{7}{4} & \frac{5}{4} & -\frac{3}{2} \\ \frac{3}{4} & -\frac{1}{4} & \frac{1}{2} \\ -\frac{5}{4} & \frac{3}{4} & -\frac{1}{2} \end{bmatrix}.$$

□

2. Compute the LU decomposition for the following matrices

(a) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 1 & 4 & 1 \end{bmatrix}$

Solution.

(a) From problem 1(a), we see that after using the multiplier $\ell_{21} = \frac{3}{2}$, we got the upper triangular matrix $U = \begin{bmatrix} 2 & 1 \\ 0 & \frac{1}{2} \end{bmatrix}$. Therefore the LU decomposition is

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

(b) Let's use elimination to get the upper triangular matrix

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{array}{l} \\ \\ r_3 - \frac{1}{2}r_3 \end{array} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

We have our upper triangular matrix, and the only multiplier required was $\ell_{31} = \frac{1}{2}$, so

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

(c) Let's use elimination to get the upper triangular matrix

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 1 & 4 & 1 \end{bmatrix} \begin{matrix} \\ r_2 + r_1 \\ r_3 - r_1 \end{matrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 2 & 2 \end{bmatrix} r_3 - \tfrac{2}{3} r_2$$
$$= \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}.$$

The multiplier that we used were $\ell_{21} = -1$, $\ell_{31} = 1$, $\ell_{32} = \frac{2}{3}$. Therefore the LU decomposition is

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}.$$

□

3. Use the LU decomposition from the previous problem to solve the following systems

(a) $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

(c) $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Solution.

We first use forward substitution to solve $L\mathbf{c} = \mathbf{b}$ then back substitution to solve $U\mathbf{x} = \mathbf{c}$.

(a) We have

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

First, we use forward subsitution to solve

$$\begin{bmatrix}1 & 0\\ \frac{3}{2} & 1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}.$$

We find that

$$\begin{aligned}c_1 &= 1\\ c_2 &= 1-\frac{3}{2}c_1=-\frac{1}{2}.\end{aligned}$$

Next, we use back subsitution to solve

$$\begin{bmatrix}2 & 1\\ 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}1\\ -\frac{1}{2}\end{bmatrix}.$$

We find that

$$\begin{aligned}x_2 &= -1\\ x_1 &= \frac{1}{2}(1-x_2)=1\end{aligned}$$

so $\mathbf{x}=\begin{bmatrix}1\\ -1\end{bmatrix}$.

(b) We have

$$\begin{bmatrix}2 & 0 & 1\\ 0 & 1 & 1\\ 1 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ \frac{1}{2} & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & \frac{1}{2}\end{bmatrix}.$$

Using forward substitution to solve

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ \frac{1}{2} & 0 & 1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\\ c_3\end{bmatrix}=\begin{bmatrix}1\\ 1\\ -1\end{bmatrix}$$

we find

$$\begin{aligned}c_1 &= 1\\ c_2 &= 1\\ c_3 &= -1-\frac{1}{2}c_1=-\frac{3}{2}.\end{aligned}$$

Now, we use back substitution to solve

$$\begin{bmatrix}2 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}1\\ 1\\ -\frac{3}{2}\end{bmatrix}.$$

We find that

$$
\begin{aligned}
x_3 &= -3 \\
x_2 &= 1 - x_3 = 4 \\
x_1 &= \frac{1}{2}(1 - x_3) = 2
\end{aligned}
$$

so $\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}$.

(c) We have

$$
\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}.
$$

First, we use forward substitution to solve

$$
\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.
$$

We find that

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= 2 + c_1 = 3 \\
c_3 &= 3 - c_1 - \frac{2}{3}c_2 = 0.
\end{aligned}
$$

Now, we use back substitution to solve

$$
\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}.
$$

We find that

$$
\begin{aligned}
x_3 &= 0 \\
x_2 &= \frac{1}{3}(3 - x_3) = 1 \\
x_1 &= 1 - 2x_2 + x_3 = -1
\end{aligned}
$$

so $\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

□

4. Use the inverse matrices found in problem 1 to find the solution to the following systems

   (a) $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   (b) $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

   (c) $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

   Solution.

   (a)

$$\mathbf{x} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

   (b)

$$\mathbf{x} = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 3 \end{bmatrix}.$$

   (c)

$$\mathbf{x} = \begin{bmatrix} -\frac{7}{4} & \frac{5}{4} & -\frac{3}{2} \\ \frac{3}{4} & -\frac{1}{4} & \frac{1}{2} \\ -\frac{5}{4} & \frac{3}{4} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{15}{4} \\ \frac{7}{4} \\ -\frac{5}{4} \end{bmatrix}.$$

□

5. Show that $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$ for the following matrices

   (a) $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 4 & 1 \\ -1 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

   Solution.

We have

$$AB = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 5 & 8 \\ 9 & 14 \end{bmatrix}$$

so $(AB)^T = \begin{bmatrix} 5 & 9 \\ 8 & 14 \end{bmatrix}$. Next,

$$B^{\mathrm{T}} A^{\mathrm{T}} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 5 & 9 \\ 8 & 14 \end{bmatrix}.$$

We see that indeed $(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}}$.

We have

$$AB = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 3 & 5 \\ 3 & 3 & 3 \\ 1 & 4 & 7 \end{bmatrix}$$

so $(AB)^{\mathrm{T}} = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 3 & 4 \\ 5 & 3 & 7 \end{bmatrix}$. Next,

$$B^T A^T = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 3 & 1 \\ 3 & 3 & 4 \\ 5 & 3 & 7 \end{bmatrix}.$$

We see that indeed $(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}}$.

We have

$$AB = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 4 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & -1 & 1 \\ 4 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix}$$

so $(AB)^{\mathrm{T}} = \begin{bmatrix} 2 & 4 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. Next,

$$\begin{aligned} B^{\mathrm{T}}A^{\mathrm{T}} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 2 & 4 & 1 \\ -1 & 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix}. \end{aligned}$$

We see that indeed $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$. □

6. Let $A \in \mathbb{R}^{n \times n}$. Show that $A^{\mathrm{T}}A$ is a symmetric matrix.

Solution.
Let $D = A^{\mathrm{T}}$. Then $d_{ij} = a_{ji}$. Now, we see

$$\begin{aligned} (A^{\mathrm{T}}A)_{ij} &= DA \\ &= \sum_{k=1}^{n} d_{ik}a_{kj} \\ &= \sum_{k=1}^{n} a_{ki}a_{kj}. \end{aligned}$$

$A^{\mathrm{T}}A$ is symmetric if $(A^{\mathrm{T}}A)_{ij} = (A^{\mathrm{T}}A)_{ji}$. We see that

$$\begin{aligned} (A^{\mathrm{T}}A)_{ji} &= \sum_{k=1}^{n} d_{jk}a_{ki} \\ &= \sum_{k=1}^{n} a_{kj}a_{ki} \\ &= \sum_{k=1}^{n} a_{ki}a_{kj} \\ &= \sum_{k=1}^{n} d_{ik}a_{kj} \\ &= (A^{\mathrm{T}}A)_{ij} \end{aligned}$$

so $A^{\mathrm{T}}A$ is indeed symmetric. □

7. Let $E_{ij} \in \mathbb{R}^{n \times n}$ be the elimination matrix

$$E_{ij} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & -\ell_{ij} & \ddots & 0 \\ & & & 1 \end{bmatrix}$$

where ℓ_{ij} is a multiplier in position (i, j). Show that

$$E_{ij}^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & \ell_{ij} & \ddots & 0 \\ & & & 1 \end{bmatrix}$$

where ℓ_{ij} is in position (i, j).

Solution.

Here are two ways we can show that the given matrix E_{ij}^{-1} is indeed the inverse.

(a) We will consider $E_{ij}\mathbf{x} = \mathbf{b}$ to find $\mathbf{x} = E_{ij}^{-1}\mathbf{b}$. By considering $E_{ij}\mathbf{x} = \mathbf{b}$, we see that

$$\begin{aligned} x_1 &= b_1 \\ x_2 &= b_2 \\ &\vdots \\ x_{i-1} &= b_{i-1} \\ -\ell_{ij}x_j + x_i &= b_i \\ x_{i+1} &= b_{i+1} \\ &\vdots \\ x_n &= b_n. \end{aligned}$$

We know that $x_j = b_j$, so $x_i = \ell_{ij}b_j + b_i$. Now, finding a matrix so that $\mathbf{x} = E_{ij}^{-1}\mathbf{b}$, we see that indeed E_{ij}^{-1} is the matrix given above.

(b) Alternatively, we can show that $E_{ij}^{-1}E_{ij} = E_{ij}E_{ij}^{-1} = I$.

□