MAT 22A Problem Set 5 Solutions

1. Let
$$P_n = \left\{ f : f = \sum_{\alpha=0}^{n} c_\alpha x^\alpha, \quad c_\alpha \in \mathbb{R} \right\}$$
be the set of polynomials of degree n. Show that P_n is a vector space.

Solution.

1) The scalar field is the real numbers.
2) The set we are considering is P_n, the polynomials of degree n.
3) Vector addition is the usual addition of functions that we are familiar with. The sum of two polynomials produces a polynomial. Let us check the remaining properties. Let f, g and h be n-degree polynomials. The following properties follow from the usual addition of real numbers that we are familiar with.
   (a) Commutativity: $f + g = g + f$
   (b) Associativity: $f + (g + h) = (f + g) + h$
   (c) Zero vector, $\mathbf{0} = 0$, $f + 0 = f$
   (d) Additive inverse, $-f(x)$, $f(x) + -f(x) = 0$
4) Scalar multiplication is the usual multiplication of a scalar and a function that we are familiar with. The product of a scalar and a polynomial is a polynomial. Let us check the remaining properties. Let f and g be n-degree polynomials and $c, d \in \mathbb{R}$. The following properties follow from the usual multiplication that we are familiar with.
   (a) $1f(x) = f(x)$
   (b) $c(df(x)) = (cd)(f(x)$
   (c) $c(f(x) + g(x)) = cf(x) + cg(x)$
   (d) $(c + d)f(x) = cf(x) + df(x)$

□

2. Let V be the space of real-valued functions. A function is odd if $f(-x) = -f(x)$ and even if $f(-x) = f(x)$. Let W be the set of odd real-valued functions and X the set of even real valued functions. Is W a subspace of V? Is X and subspace of V?

Solution.

We need to show that (a) vector addition of elements in the subspace stay in the subspace and (b) scalar multiplication of an element of the subspace stays in the subspace (this also shows that the zero vector is in the subspace).

First, we consider the set of even real valued functions. Let f and g be even functions and let $h(x) = f(x) + g(x)$. Then

$$h(-x) = f(-x) + g(-x) = f(x) + g(x) = h(x)$$

so $h(-x) = h(x)$, so $h(x)$ is an even function. So addition of two even functions produces an even function so (a) is satisfied. Let $k(x) = cf(x)$ where $c \in \mathbb{R}$. We see that

$$k(-x) = cf(-x) = cf(x) = k(x)$$

so $k(x)$ is an even function so (b) is satisfied. Also, $0f(x) = 0$, which is an even function.

Next, we cosider the set of odd real-valued functions. Let f and g be odd functions and let $h(x) = f(x) + g(x)$. Then

$$h(-x) = f(-x) + g(-x) = -f(x) + -g(x) = -(f(x) + g(x)) = -h(x)$$

so we see that (a) is satisfied. Next, let $k(x) = cf(x)$ where $c \in \mathbb{R}$. We see that

$$k(-x) = cf(-x) = c(-f(x)) = -cf(x) = -k(x)$$

so $k(x)$ is an odd function. Also, $0f(x) = 0$ which is an odd function.

□

3. Let V be the set of pairs (x, y) with $x, y \in \mathbb{R}$. Define

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, 0)$$
$$c(x_1, y_1) = (cx_1, 0)$$

where $c \in \mathbb{R}$. Is V with these operations a vector space?

Solution.

We want to check that the conditions required of vector addition and scalar multiplication are satisfied. When we check to see if there is a vector $\mathbf{0}$, so that $(x, y) + \mathbf{0} = (x, y)$, we see that we cannot find such a vector. Let $\mathbf{0} = (0_1, 0_2)$. We see that

$$(x, y) + \mathbf{0} = (x + 0_1, 0).$$

We see that there is no way to recover the second entry of the pair (x, y). Therefore, there is no appropriate zero vector, so V with the defined operations does not form a vector space. □

4. Show that the space of real diagonal matrices form a vector space.

Solution.

1) The scalar field is the real numbers.

2) The set we are considering the the set of real diagonal matrices, so the vectors are real diagonal matrices.

3) Vector additional is the familiar entry-wise matrix addition. The sum of two real diagonal matrices produces a real diagonal matrix. Let A, B, and C be $n \times n$ real diagonal matrices. We verify the following properties

   (a) Commutativity:

$$\begin{aligned}
A + B &= \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} + \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{bmatrix} \\
&= \begin{bmatrix} a_1 + b_1 & & \\ & \ddots & \\ & & a_n + b_n \end{bmatrix} \\
&= \begin{bmatrix} b_1 + a_1 & & \\ & \ddots & \\ & & b_n + a_n \end{bmatrix} \\
&= \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & ba_n \end{bmatrix} + \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \\
&= B + A.
\end{aligned}$$

   (b) Associativity:

$$\begin{aligned}
A + (B + C) &= \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} + \left(\begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{bmatrix} + \begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{bmatrix} \right) \\
&= \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} + \begin{bmatrix} b_1 + c_1 & & \\ & \ddots & \\ & & b_n + c_n \end{bmatrix} \\
&= \begin{bmatrix} a_1 + (b_1 + c_1) & & \\ & \ddots & \\ & & a_n + (b_n + c_n) \end{bmatrix} \\
&= \begin{bmatrix} (a_1 + b_1) + c_1 & & \\ & \ddots & \\ & & (a_n + b_n) + c_n \end{bmatrix} \\
&= (A + B) + C.
\end{aligned}$$

(c) The zero vector of the zero matrix which has entires which are all zero. The zero matrix is a real diagonal matrix, and we see that

$$A + \mathbf{0} = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} + \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix} = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} = A.$$

(d) We also see that for any real diagonal matrix A, there is a matrix $-A$ so that $A + -A = \mathbf{0}$. For uniqueness, suppose there is a matrix B so that $A + B = \mathbf{0}$. Then, $a_i + b_i = 0$ and $a_i = -b_i$, so $B = -A$.

4) Scalar multiplication multiplies each of the entires of the matrix by the scalar. Clearly multiplying a diagonal matrix by a scalar produces a diagonal matrix. Let A and B be a $n \times n$ real diagonal matrix and $c, d \in \mathbb{R}$. The following properties clearly follow from out previous discussion of matrix operations:

(a) Clearly $1A = A$.
(b) $c(dA) = (cd)A$
(c) $c(A + B) = cA + cB$
(d) $(c + d)A = cA + dA$

$\square$

5. For the following matrices (a) Find the nullspace of A and (b) Find the rank of A.

(i) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & 0 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 2 & -1 & 0 & 1 & -1 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 3 & 2 & -1 \end{bmatrix}$

Solution.

(a) First, we find rref(A):

$$\begin{aligned} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix} r_2 - 2r_1 &= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} -\tfrac{1}{3} r_2 \\ &= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} r_1 - 2r_2 \\ &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \\ &= R. \end{aligned}$$

Considering $R\mathbf{x} = \mathbf{0}$, we see that $x_1 = x_3$ and $x_2 = -2x_3$, so

$$\mathbf{x} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

So the nullspace of A is spanned by $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. From R we see that the rank of A is 2.

(b) First, we find rref(A):

$$\begin{aligned}
\begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{matrix} \\ r_2 - 3r_1 \\ r_3 - r_1 \end{matrix} &= \begin{bmatrix} 1 & 2 \\ 0 & -4 \\ 0 & -2 \end{bmatrix} \begin{matrix} r_1 + \frac{1}{2}r_2 \\ \\ r_3 - \frac{1}{2}r_2 \end{matrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & -4 \\ 0 & 0 \end{bmatrix} \begin{matrix} \\ -\frac{1}{4}r_2 \\ \\ \end{matrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \\
&= R.
\end{aligned}$$

Consider $R\mathbf{x} = \mathbf{0}$, we see that $x_1 = 0$ and $x_2 = 0$. x_3 is a free variable, so we see that the nullspace of A is spanned by $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. From R, we see that the rank of A is 2.

(c) First, we find rref(A):

$$\begin{aligned}
\begin{bmatrix} 1 & 2 & -1 & 0 & 1 & -1 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 3 & 2 & -1 \end{bmatrix} \begin{matrix} \\ r_2 - r_1 \\ \\ \end{matrix} &= \begin{bmatrix} 1 & 2 & -1 & 0 & 1 & -1 \\ 0 & 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 2 & 3 & 2 & -1 \end{bmatrix} \begin{matrix} r_1 + \frac{1}{3}r_2 \\ \\ r_3 - \frac{2}{3}r_2 \end{matrix} \\
&= \begin{bmatrix} 1 & 2 & 0 & 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 & 2 & -\frac{5}{3} \end{bmatrix} \begin{matrix} \\ \frac{1}{3}r_2 \\ \frac{1}{3}r_3 \end{matrix} \\
&= \begin{bmatrix} 1 & 2 & 0 & 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & \frac{2}{3} & -\frac{5}{9} \end{bmatrix} \\
&= R.
\end{aligned}$$

Now, considering $R\mathbf{x} = \mathbf{0}$, we see that $x_1 = -2x_2 - x_5 + \frac{2}{3}x_6$, $x_3 =$

$-\frac{1}{3}x_6$, and $x_4 = -\frac{2}{3}x_5 + \frac{5}{9}x_6$. We see that

$$\mathbf{x} = \begin{bmatrix} -2x_2 - x_5 + \frac{2}{3}x_6 \\ x_2 \\ -\frac{1}{3}x_6 \\ -\frac{2}{3}x_5 + \frac{5}{9}x_6 \\ x_5 \\ x_6 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} \frac{2}{3} \\ 0 \\ -\frac{1}{3} \\ \frac{5}{9} \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, the nullspace is span$\left(\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ 0 \\ -\frac{1}{3} \\ \frac{5}{9} \\ 0 \\ 1 \end{bmatrix} \right\}\right)$. From R, we see that the rank of A is 3.

□