MAT 22A Problem Set 6 Solutions

1. Find the complete solution to

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \\ 2 \end{bmatrix}.$$

Write the complete solution as $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$.

Solution.

First we row reduce the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & -3 \\ 2 & 6 & 4 & 8 & -4 \\ 0 & 0 & 2 & 4 & 2 \end{array}\right] r_2 - 2r_1 = \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & -3 \\ 0 & 0 & 2 & 4 & 2 \\ 0 & 0 & 2 & 4 & 2 \end{array}\right] \begin{array}{c} r_1 - \frac{1}{2}r_2 \\ \\ r_3 - r_2 \end{array}$$

$$= \left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & -4 \\ 0 & 0 & 2 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \frac{1}{2}$$

$$= \left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & -4 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We see that $\mathbf{x}_p = \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{x}_n = \begin{bmatrix} -3x_2 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$. Thus, the complete solution is

$$\mathbf{x} = \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

□

2. Give an example of a matrix A for each of the four possibilities for linear equations depending on the rank r. Show that the matrix has the corresponding number of solutions.

Solution.

Let A be an $m \times n$ matrix, and $\text{rank}(A) = r$.

(a) If $r = m = n$, A is an invertible matrix, so $A\mathbf{x} = \mathbf{b}$ has only one solution. For example, consider $A = I$.

(b) If $r = m < n$, we can consider the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Then

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x} = \mathbf{b}$$

has infinitely many solutions of the form $\mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(c) If $r = n < m$, we can consider the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{b}$$

has solution $\mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ if $b_3 = 0$. If $b_3 \neq 0$, then the system has no solution.

(d) If $r < m$ and $r < n$, we can consider the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{b}$$

has infinitely many solutions of the form $\mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ if $b_3 = 0$. If $b_3 \neq 0$, then the system has no solution.

$\square$

3. Find a basis for the column space and nullspace of A. What is the dimension of the nullspace and columnspace? What is the rank of A?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix}.$$

Solution.

First, we find rref(A).

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix} \begin{matrix} \\ r_2 - r_1 \\ r_3 - 3r_1 \end{matrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix} \begin{matrix} r_1 - \frac{1}{2}r_2 \\ \\ r_3 + r_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \frac{1}{2}r_2$$

$$= \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

We see that a basis of the column space is given by the first two columns of A. Then $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ form a basis of the column space of A. For the nullspace, we solve $A\mathbf{x} = \mathbf{0}$, so $\mathbf{x} = \begin{bmatrix} \frac{1}{2}x_3 \\ -\frac{1}{2}x_3 \\ x_3 \end{bmatrix}$, so $\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$ forms a basis of $\mathrm{N}(A)$. We see that the column space is 2-dimensional and $\mathrm{N}(A)$ is 1-dimensional. The rank of A is the dimension of the column space (=dimension of the row space = number of pivots), so $\mathrm{rank}(A) = 2$. □

4. Suppose $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are linearly independent.

   (a) Let $\mathbf{v}_1 = \mathbf{u}_2 - \mathbf{u}_3$, $\mathbf{v}_2 = \mathbf{u}_1 - \mathbf{u}_3$, and $\mathbf{v}_3 = \mathbf{u}_1 - \mathbf{u}_2$. Determine whether $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent.

   (b) Let $\mathbf{v}_1 = \mathbf{u}_2 + \mathbf{u}_3$, $\mathbf{v}_2 = \mathbf{u}_1 + \mathbf{u}_3$, and $\mathbf{v}_3 = \mathbf{u}_1 + \mathbf{u}_2$. Determine whether $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent.

Solution.

(a) $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

requires $c_1 = c_2 = c_3 = 0$. We see that

$$\begin{aligned} \mathbf{0} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \\ &= c_1(\mathbf{u}_2 - \mathbf{u}_3) + c_2(\mathbf{u}_1 - \mathbf{u}_3) + c_3(\mathbf{u}_1 - \mathbf{u}_2) \\ &= (c_2 + c_3)\mathbf{u}_1 + (c_1 - c_3)\mathbf{u}_2 + (-c_1 - c_2)\mathbf{u}_3. \end{aligned}$$

Since $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are linearly independent, so

$$\begin{aligned} c_2 + c_3 &= 0 \\ c_1 - c_3 &= 0 \\ -c_1 - c_2 &= 0 \end{aligned}$$

and we have that $c_2 = -c_3$, $c_1 = c_3$ and $c_2 = -c_1$. Letting $c_1 = 1$, we see that $c_2 = -1$, and $c_3 = 1$. Then, we see that

$$\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$$

so $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are not linearly independent.

(b) Again, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

requires $c_1 = c_2 = c_3 = 0$. We see that

$$\begin{aligned}\mathbf{0} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \\ &= c_1(\mathbf{u}_2 + \mathbf{u}_3) + c_2(\mathbf{u}_1 + \mathbf{u}_3) + c_3(\mathbf{u}_1 + \mathbf{u}_2) \\ &= (c_2 + c_3)\mathbf{u}_1 + (c_1 + c_3)\mathbf{u}_2 + (c_1 + c_2)\mathbf{u}_3.\end{aligned}$$

Since $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ are linearly independent,

$$\begin{aligned}c_2 + c_3 &= 0 \\ c_1 + c_3 &= 0 \\ c_1 + c_2 &= 0\end{aligned}$$

So $c_2 = -c_3$, $c_1 = -c_3$ and $c_1 = -c_2$. Then we must have that $c_2 = c_1$ from the first two relations and so $c_1 = -c_1$. Then $c_1 = 0$, so $c_2 = 0$ and $c_3 = 0$. Thus, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent.

□