MAT 22A Problem Set 7 (Due 7/21 5 PM)

1. Find a basis and determine the dimension of the four fundamental subspaces for the following matrices

   (a) $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \end{bmatrix}$.

   (b) $A = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 2 & 6 & 3 & 4 \\ 1 & 3 & 3 & 8 \end{bmatrix}$.

2. Find the dimension of each of the four fundamental subspaces for the following matrices

   (a) A is a 5×20 matrix and rank(A)=3.

   (b) A is a 4000×100 matrix and rank(A) $= 15$.

3. Prove that if rank(A)=r, then A is the sum of r rank 1 matrices.

4. Let

$$\begin{aligned} a_1 &= (1,-1) & b_1 &= (1,0) \\ a_2 &= (2,-1) & b_2 &= (0,1) \\ a_3 &= (-3,2) & b_3 &= (1,1). \end{aligned}$$

   Is there a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$ so that $Ta_i = b_i$ for $i = 1, 2, 3$? If so, find T.

5. Prove that the following are linear transformations or provide an example that shows that it is not a linear transformation. If it is a linear transformation, describe the space the transformation maps to, and describe the vectors in the nullspace.

   (a) Let V be the vector space $\mathbb{R}^2$. $T(x_1, x_2) = (x_2, x_1)$.

   (b) Let V be the vector space $\mathbb{R}^2$. $T(x_1, x_2) = (\cos(x_1), x_2)$.

   (c) Let V be the vector space $\mathbb{R}^n$, A an $m \times n$ matrix and $\mathbf{b}$ an $m \times 1$ column vector. $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$.

   (d) Let V be the vector space of $n \times n$ real matrices and B a fixed $n \times n$ matrix. $T(A) = AB - BA$.

   (e) Let V be the vector space of real continuous functions. $(Tf)(x) = \int_a^x f(t)\,dt$.

   (f) Let V be the vector space $\mathbb{R}^n$. $T(\mathbf{v}) = \|\mathbf{v}\|$.

   (g) Let V be the vector space of n-degree polynomials. $(Tf)(x) = \frac{d^2}{dx^2} f(x)$.