MAT 22A Problem Set 7 Solutions

1. Find a basis and determine the dimension of the four fundamental subspaces for the following matrices

   (a) $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \end{bmatrix}$.

   (b) $A = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 2 & 6 & 3 & 4 \\ 1 & 3 & 3 & 8 \end{bmatrix}$.

   Solution.

   (a) First we find rref(A).

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \end{bmatrix} \begin{matrix} \\ r_2 - 2r_1 \end{matrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & -7 \end{bmatrix} \begin{matrix} r_1 + \frac{7}{4}r_2 \\ \end{matrix}$$
$$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -7 \end{bmatrix} \begin{matrix} \\ -\frac{1}{7}r_2 \end{matrix}$$
$$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   The pivot columns of A form a basis of $C(A)$ so $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ form a basis of $C(A)$, so $C(A)$ is 2-dimensional. The rows of rref(A) and the pivot rows of A form a basis of $C(A^{\mathrm{T}})$ (the row space), so $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis of $C(A^{\mathrm{T}})$. We see that $C(A^{\mathrm{T}})$ is 2-dimensional. Note that $\mathrm{rank}(A) = \dim(\mathrm{C}(A)) = \dim(\mathrm{C}(A^{\mathrm{T}}))$. Now, the nullspace of A is found by solving $\mathrm{rref}(A)\mathbf{x} = 0$. We find that a basis of $\mathrm{N}(A)$ is $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, and so the nullspace is 1-dimensional. Finally, A has full row rank, so the left nullspace is 0-dimensional.

(b) First we find rref(A).

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 2 & 6 & 3 & 4 \\ 1 & 3 & 3 & 8 \end{bmatrix} \begin{matrix} \\ r_2 - 2r_1 \\ r_3 - r_1 \end{matrix} = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & 4 \end{bmatrix} \begin{matrix} r_1 + 2r_2 \\ \\ r_3 + r_2 \end{matrix}$$
$$= \begin{bmatrix} 1 & 3 & 0 & -4 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ -r_2 \\ \\ \end{matrix}$$
$$= \begin{bmatrix} 1 & 3 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that C(A) is 2-dimensional and a basis of C(A) is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$. C($A^{\mathrm{T}}$) is 2-dimensional and a basis of C(A^{T}) is $\begin{bmatrix} 1 \\ 3 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 3 \\ 4 \end{bmatrix}$. The nullspace is 2-dimensional and a basis of N(A) is $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -4 \\ 1 \end{bmatrix}$. The left nullspace is 1-dimensional and a basis of $N(A^{\mathrm{T}})$ is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

□

2. Find the dimension of each of the four fundamental subspaces for the following matrices

   (a) A is a 5×20 matrix and rank(A)=3.

   (b) A is a 4000×100 matrix and rank(A) = 15.

   Solution.

   (a) Since rank(A) = 3, the dimension of the column space and row space is 3. By the rank-nullity theorem, the dimension of the nullspace is $20 - 3 = 17$, and the dimension of the left nullspace is $5 - 3 = 2$.

(b) Since $\text{rank}(A) = 15$, the dimension of the column space and row space is 15. By the rank-nullity theorem, the dimension of the nullspace is $100 - 15 = 85$, and the dimension of the left nullspace is $4000 - 15 = 3985$.

□

3. Prove that if rank(A)=r, then A is the sum of r rank 1 matrices.

Solution.

Since A is a rank r matrix, we can row reduce A using a series of matricies, so $A = CR$ where $R = \text{rref}(A)$. Recall that we can compute the matrix-matrix product of an $m \times n$ matrix A and an $n \times k$ matrix B by summing up n rank 1 matrices. That is,

$$AB = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} \mathbf{b}_1^{\mathrm{T}} \\ \vdots \\ \mathbf{b}_n^{\mathrm{T}} \end{bmatrix} = \mathbf{a}_1\mathbf{b}_1^{\mathrm{T}} + \cdots + \mathbf{a}_n\mathbf{b}_n^{\mathrm{T}}.$$

Now, since $\text{rank}(A) = r$, A has $m - r$ rows of zeros. Then we have

$$A = CR = \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_n \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \vdots \\ \mathbf{v}_r^{\mathrm{T}} \\ \mathbf{0}^{\mathrm{T}} \\ \vdots \\ \mathbf{0}^{\mathrm{T}} \end{bmatrix} = \mathbf{c}_1\mathbf{v}_1^{\mathrm{T}} + \cdots + \mathbf{c}_r\mathbf{v}_r^{\mathrm{T}}$$

□

4. Let

$$\begin{aligned} a_1 &= (1, -1) & b_1 &= (1, 0) \\ a_2 &= (2, -1) & b_2 &= (0, 1) \\ a_3 &= (-3, 2) & b_3 &= (1, 1). \end{aligned}$$

Is there a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$ so that $Ta_i = b_i$ for $i = 1, 2, 3$? If so, find T.

Solution.

We want to find a 2×2 matrix T so that $Ta_i = b_i$. We have

$$\begin{aligned} t_{11} - t_{12} &= 1 \\ t_{21} - t_{22} &= 0 \\ 2t_{11} - t_{12} &= 0 \\ 2t_{21} - t_{22} &= 1 \end{aligned}$$

so $t_{11} = -1$, $t_{12} = -2$, $t_{21} = 1$, and $t_{22} = 1$. Then

$$T = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

but

$$\begin{aligned} Ta_3 &= \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ -1 \end{bmatrix} \\ &\neq \begin{bmatrix} 1 \\ 1 \end{bmatrix} = b_3. \end{aligned}$$

We see that we have an over-determined system since we have 6 equations for 4 variables. Notice that $a_3 = -a_1 - a_2$. Then

$$T(a_3) = T(-a_1 - a_2) = -T(a_1) - T(a_2) = -(1,0) - (0,1) = (-1,-1) \neq (1,1) = b_3$$

so we cannot find a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ so that $Ta_i = b_i$.

$\square$

5. Prove that the following are linear transformations or provide an example that shows that it is not a linear transformation. If it is a linear transformation, describe the space the transformation maps to, and describe the vectors in the nullspace.

   (a) Let V be the vector space $\mathbb{R}^2$. $T(x_1, x_2) = (x_2, x_1)$.

   (b) Let V be the vector space $\mathbb{R}^2$. $T(x_1, x_2) = (\cos(x_1), x_2)$.

   (c) Let V be the vector space $\mathbb{R}^n$, A an $m \times n$ matrix and $\mathbf{b}$ an $m \times 1$ column vector. $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$.

   (d) Let V be the vector space of $n \times n$ real matrices and B a fixed $n \times n$ matrix. $T(A) = AB - BA$.

   (e) Let V be the vector space of real continuous functions. $(Tf)(x) = \int_a^x f(t)\,dt$.

   (f) Let V be the vector space $\mathbb{R}^n$. $T(\mathbf{v}) = \|\mathbf{v}\|$.

   (g) Let V be the vector space of n-degree polynomials. $(Tf)(x) = \frac{d^2}{dx^2} f(x)$.

   Solution.

(a) Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ and $a \in \mathbb{R}$. Then

$$\begin{aligned} T(a\mathbf{v}+\mathbf{w}) &= T(av_1+w_1, av_2+w_2) \\ &= (av_2+w_2, av_1+w_1) \\ &= (av_2, v_1) + (w_2, w_1) \\ &= a(v_2, v_1) + (w_2, w_1) \\ &= aT\mathbf{v} + T\mathbf{w} \end{aligned}$$

and so we see that T is indeed a linear transformation. T produces 2-tuples, so T maps to $\mathbb{R}^2$. The nullspace of T is the collection of $\mathbf{x}$ so that $T(x_1, x_2) = (0,0)$. We see that $T(x_1, x_2) = (x_2, x_1) = (0,0)$, so the nullspace of T is just $\mathbf{0}$.

(b) T is not a linear transformation. Consider $(0,1)$ and $(\pi, 1)$.

$$T(0+\pi, 1+1) = (\cos(0+\pi), 2) = (-1, 2)$$

but $T(0,1) = (1,1)$ and $T(\pi, 1) = (-1, 1)$. We see that

$$T(0+\pi, 1+1) \neq T(0,1) + T(\pi, 1).$$

(c) Not a linear transformation if $\mathbf{b} \neq \mathbf{0}$. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $a \in \mathbb{R}$. Then

$$\begin{aligned} T(a\mathbf{v}+\mathbf{w}) &= A(a\mathbf{v}+\mathbf{w}) + \mathbf{b} \\ &= aA\mathbf{v} + A\mathbf{w} + \mathbf{b} \\ &= aA\mathbf{v} + \mathbf{b} + A\mathbf{w} + \mathbf{b} - \mathbf{b} \\ &= aT(\mathbf{v}) + T(\mathbf{w}) - \mathbf{b} \end{aligned}$$

(d) Let A and C be $n \times n$ real matrices and $c \in \mathbb{R}$. Then

$$\begin{aligned} T(cA+C) &= (cA+C)B - B(cA+C) \\ &= cAB + CB - cBA - BC \\ &= c(AB - BA) + CB - BC \\ &= cT(A) + T(B) \end{aligned}$$

and so T is a linear transformation. T produces $n \times n$ matrices so T maps to the set of $n \times n$ matrices. For $T(A) = 0$, we see that $AB = BA$, so the nullspace of T is the collection of matrices that commute with B.

(e) Let f and g be real continuous functions and $c \in \mathbb{R}$. Using what we know from calculus, we see that

$$
\begin{aligned}
(T(cf+g))(x) &= \int_a^x cf(t) + g(t)\, dt \\
&= \int_a^x cf(t)\, dt + \int_a^t g(t)\, dt \\
&= c\int_a^x f(t)\, dt + \int_a^t g(t)\, dt \\
&= c(Tf)(x) + (Tg)(x)
\end{aligned}
$$

so T is a linear transformation. T produces a continuous function, so T maps to the space of real continuous functions. Let $F(x)$ be the anti-derivative of $f(x)$. Then

$$
\begin{aligned}
(Tf)(x) &= \int_a^x f(t)\, dt \\
&= F(x) - F(a)
\end{aligned}
$$

so for $(Tf)(x) = 0$, we need $F(x) = F(a)$ for all x. Then, $F(x)$ is a constant function. Now, $f(x) = F'(x) = 0$ since F is constant. Therefore, the nullspace of T only contains the zero function.

(f) Not a linear transformation. Consider $\mathbf{v} = (1, 1, \ldots, 1)$ and $\mathbf{w} = (1, 0, \ldots, 0)$. Then

$$
\begin{aligned}
T(\mathbf{v}+\mathbf{w}) &= T((2, 1, \ldots, 1)) \\
&= \|(2, 1, \ldots, 1)\| \\
&= \sqrt{4 + \sum_{i=2}^{n} 1} \\
&= \sqrt{3+n}
\end{aligned}
$$

but $T(\mathbf{v}) = \sqrt{n}$ and $T(\mathbf{w}) = 1$. We see that $T(\mathbf{v}+\mathbf{w}) \neq T(\mathbf{v}) + T(\mathbf{w})$.

(g) Let f and g be n-degree polynomials and $a \in \mathbb{R}$. Using what we know from calculus, we see that

$$
\begin{aligned}
(T(af+g))(x) &= \frac{d^2}{dx^2}(af+g) \\
&= \frac{d^2}{dx^2}af + \frac{d^2}{dx^2}g \\
&= a\frac{d^2}{dx^2}f + \frac{d^2}{dx^2}g \\
&= a(Tf)(x) + (Tg)(x)
\end{aligned}
$$

and so T is a linear transformation. Next, to determine the nullspace of T, we want to find f so that $(Tf)(x) = 0$. We see that

$$\begin{aligned}(Tf)(x) &= 0 \\ \frac{d^2}{dx^2} f(x) &= 0 \\ \frac{d}{dx} f(x) &= a \\ f(x) &= ax + b\end{aligned}$$

and so functions of the form $f(x) = ax + b$ are in the nullspace of T. Finally, an n-degree polynomial may be written as $\sum_{i=0}^{n} c_i x^i$. We see that

$$\begin{aligned}(T\sum_{i=0}^{n} c_i x^i)(x) &= \frac{d^2}{dx^2} \sum_{i=0}^{n} c_i x^i \\ &= \frac{d}{dx} \sum_{i=1}^{n} c_i i x^{i-1} \\ &= \sum_{i=2}^{n} c_i i(i-1) x^{i-2} \\ &= \sum_{j=0}^{n-2} c_{j+2}(j+2)(j+1)x^j\end{aligned}$$

and so T maps to the space of $(n-2)$-degree polynomials.

□