MAT 22A Problem Set 8 Solutions

1. Let $S = \text{span}\left(\begin{bmatrix}1\\2\\2\\-1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\2\\2\\1\end{bmatrix}, \begin{bmatrix}0\\2\\-1\\2\\1\end{bmatrix}\right)$. Find a basis for $S^\perp$ and determine the dimension of $S^\perp$.

Solution.

Let A be the matrix whose columns are the vectors $\left\{\begin{bmatrix}1\\2\\2\\-1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\2\\2\\1\end{bmatrix}, \begin{bmatrix}0\\2\\-1\\2\\1\end{bmatrix}\right\}$.

Then $\mathrm{C}(A) = S$, so $S^\perp = \mathrm{C}(A)^\perp = \mathrm{N}(A^\mathrm{T})$. That is $S^\perp$ is the left nullspace of A. Now, we find rref(A).

$$\begin{bmatrix}1 & -1 & 0\\2 & 1 & 2\\2 & 2 & -1\\-1 & 2 & 2\\0 & 1 & 1\end{bmatrix}\begin{matrix}\\r_2-2r_1\\r_3-2r_1\\r_4+r_1\\ \\ \end{matrix} = \begin{bmatrix}1 & -1 & 0\\0 & -1 & 2\\0 & 4 & -1\\0 & 1 & 2\\0 & 1 & 1\end{bmatrix}\begin{matrix}r_1-r_2\\ \\r_3+4r_2\\r_4+r_2\\r_5+r_2\end{matrix}$$
$$= \begin{bmatrix}1 & 0 & -2\\0 & -1 & 2\\0 & 0 & -7\\0 & 0 & 4\\0 & 0 & 3\end{bmatrix}\begin{matrix}\\-r_2\\-\frac{1}{7}r_3\\\frac{1}{4}r_4\\\frac{1}{3}r_5\end{matrix}$$
$$= \begin{bmatrix}1 & 0 & -2\\0 & 1 & -2\\0 & 0 & 1\\0 & 0 & 1\\0 & 0 & 1\end{bmatrix}\begin{matrix}r_1+2r_3\\r_2+2r_3\\ \\r_4-r_3\\r_5-r_3\end{matrix}$$
$$= \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}.$$

Now, we solve $\mathbf{x}^\mathrm{T}R = \mathbf{0}$. We see that $x_1 = x_2 = x_3 = 0$ and x_4, x_5 are free variables. Then, $\left\{\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right\}$ forms a basis of $S^\perp$, so $S^\perp$ is 2-dimensional. □

2. Find the nullspace of an $n \times n$ invertible matrix A and determine the dimension of $\mathrm{N}(A)$. Explain your reasoning.

   Solution.

   There is a number of ways to approach this problem. Since A is invertible, A^{-1} exists, so $A\mathbf{x} = \mathbf{0}$ only has the solution $\mathbf{x} = \mathbf{0}$. Then $\mathrm{N}(A)$ is 0-dimensional.

   Another line of reasoning is as follows. Since A is invertible, the columns of A of linearly independent, so $\mathrm{rank}(A) = n$. Then $\mathrm{C}(A)$ is n-dimensional. By the rank-nullity theorem, we find that $\mathrm{nullity}(A) = n - \mathrm{rank}(A) = n - n = 0$.

   □

3. Find the pseudoinverse of the following matrix, and compute the product of the pseudoinverse and given matrix.

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

Solution.

First, we find rref(A).

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} \\ r_2 - 2r_1 \\ r_3 - 2r_1 \\ r_4 + r_1 \\ \\ \end{matrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 2 \\ 0 & 4 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} r_1 + \frac{1}{3}r_2 \\ \\ r_3 - \frac{4}{3}r_2 \\ r_4 - \frac{1}{3}r_2 \\ r_5 - \frac{1}{3}r_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 3 & 2 \\ 0 & 0 & -\frac{11}{3} \\ 0 & 0 & \frac{4}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{matrix} \\ \\ -\frac{3}{11}r_3 \\ \\ \\ \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & \frac{4}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{matrix} r_1 - \frac{2}{3}r_3 \\ r_2 - 2r_3 \\ \\ r_4 - \frac{4}{3}r_3 \\ r_5 - \frac{1}{3}r_3 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \frac{1}{3}r_2 \\ \\ \\ \\ \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We see the 3×3 identity matrix, so we take

$$B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 2 \\ 2 & 2 & -1 \end{bmatrix}.$$

Now, we find B^{-1},

$$
\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ r_2 - 2r_1 \\ r_3 - 2r_1 \end{array} = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 2 & -2 & 1 & 0 \\ 0 & 4 & -1 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} r_1 + \frac{1}{3}r_2 \\ \\ r_3 - \frac{4}{3}r_2 \end{array}
$$

$$
= \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 3 & 2 & -2 & 1 & 0 \\ 0 & 0 & -\frac{11}{3} & \frac{2}{3} & -\frac{4}{3} & 1 \end{array}\right] \begin{array}{l} r_1 + \frac{2}{11}r_3 \\ r_2 + \frac{6}{11}r_3 \\ \\ \end{array}
$$

$$
= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{15}{33} & \frac{3}{33} & \frac{2}{11} \\ 0 & 3 & 0 & -\frac{54}{33} & \frac{9}{33} & \frac{6}{11} \\ 0 & 0 & -\frac{11}{3} & \frac{2}{3} & -\frac{4}{3} & 1 \end{array}\right] \begin{array}{l} \\ \frac{1}{3}r_2 \\ -\frac{3}{11}r_3 \end{array}
$$

$$
= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{15}{33} & \frac{3}{33} & \frac{2}{11} \\ 0 & 1 & 0 & -\frac{18}{33} & \frac{3}{33} & \frac{2}{11} \\ 0 & 0 & 1 & -\frac{2}{11} & \frac{4}{11} & -\frac{3}{11} \end{array}\right]
$$

so

$$
B^{-1} = \begin{bmatrix} \frac{15}{33} & \frac{3}{33} & \frac{2}{11} \\ -\frac{18}{33} & \frac{1}{11} & \frac{2}{11} \\ -\frac{2}{11} & \frac{4}{11} & -\frac{3}{11} \end{bmatrix}
$$

Now, if we try to compute $B^{-1}A$, we see that is it not defined. But we may compute AB^{-1}, and we find

$$
AB^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{15}{33} & \frac{3}{33} & \frac{2}{11} \\ -\frac{18}{33} & \frac{1}{11} & \frac{2}{11} \\ -\frac{2}{11} & \frac{4}{11} & -\frac{3}{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{63}{33} & \frac{27}{33} & -\frac{12}{33} \\ -\frac{24}{33} & \frac{15}{33} & -\frac{3}{33} \end{bmatrix}
$$

Alternatively, we may compute the left pseudoinverse of A by computing $L = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$.

$\square$

4. Let S be a subspace of the vector space $\mathbb{R}^n$. Prove that $\mathbb{R}^n = S \oplus S^{\perp}$. That is, for any $\mathbf{v} \in \mathbb{R}^n$,

$$
\mathbf{v} = \mathbf{s} + \mathbf{t}
$$

where $\mathbf{s} \in S$ and $\mathbf{t} \in S^{\perp}$.

Solution.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be a basis of S. Let A be the matrix

$$
A = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k \end{bmatrix}.
$$

Then $\mathrm{C}(A) = S$, so $S^{\perp} = \mathrm{N}(A^{\mathrm{T}})$. By the rank-nullity theorem, $n =$rank$(A^{\mathrm{T}})+$ nullity(A^{T}) = rank(A)+nullity(A^{T}), so nullity$(A^{\mathrm{T}}) = n - k$. Then $S^{\perp}$

is $(n-k)$-dimensional, so let $\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n$ be a basis of $S^\perp$. We know $\mathrm{C}(A) \oplus \mathrm{N}(A^\mathrm{T}) = \mathbb{R}^n$, so $S \oplus S^\perp = \mathbb{R}^n$. We see that for any $\mathbf{v} \in \mathbb{R}^n$,

$$\mathbf{v} = \mathbf{s} + \mathbf{t}$$

where $\mathbf{s} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k \in S$ and $\mathbf{t} = c_{k+1}\mathbf{v}_{k+1} + \cdots + c_n\mathbf{v}_n \in S^\perp$. $\square$