Chapter 6 The Laplace Transform

6.1 Definition of the Laplace Transform

- Recall: Improper integral

\[ \int_a^\infty f(t) \, dt = \lim_{b \to \infty} \int_a^b f(t) \, dt \]

- If \( \int_a^b f(t) \, dt \) exists for all \( b > a \), and the limit as \( b \to \infty \) exists, then the improper integral is said to converge. Otherwise, it diverges.

Remark: Showing convergence directly may be difficult, so we use comparison tests.

- A function \( f \) is called piecewise continuous on an interval \( \alpha \leq t \leq \beta \) if the interval
can be partitioned by a finite number of points $a = t_0 < t_1 < \cdots < t_n = b$ so that

1. $f$ is continuous on each subinterval $t_{i−1} < t < t_i$

2. $f$ approaches a finite limit as each endpoint of each subinterval is approached from within the subinterval.

Remark 1. We may now write $\int_a^b f(t) \, dt$ as

$$\int_a^b f(t) \, dt = \int_{t_0}^{t_1} f(t) \, dt + \int_{t_1}^{t_2} f(t) \, dt + \cdots + \int_{t_{n−1}}^{t_n} f(t) \, dt$$
Then if \( f(t) \) is piecewise continuous for \( a \leq t \), if \( |f(t)| \leq g(t) \) when \( t \geq M \) for some positive constant, and if \( \int_M^\infty g(t) \, dt \) converges, then \( \int_a^\infty f(t) \, dt \) converges.

On the other hand, if \( f(t) \geq g(t) \geq 0 \) for \( t \geq M \) and \( \int_M^\infty g(t) \, dt \) diverges, then \( \int_a^\infty f(t) \, dt \) diverges.
The Laplace Transform

Integral Transform

\[ F(s) = \int_0^\infty k(s,t) f(t) \, dt \]

- \( k(s,t) \) is given and called the kernel of the transformation
- \( F(s) \) is called the transform of \( f(t) \)

**Def Laplace Transform**

\[ L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) \, dt \]

Here, the kernel is \( k(s,t) = e^{-st} \)

Steps for using the Laplace transform to solve a DE:
1. Transform the DE from t-domain to a simpler problem in the s-domain.
2. Solve the algebraic problem to find $F(s)$.
3. Recover $f$ from it transform $F$.

Then suppose that

(i) $f$ is piecewise continuous on $0 \leq t \leq b$ for any positive $b$.

(ii) There are constants $K$, $a$, and $M$ so that

$$|f(t)| \leq Ke^{at} \quad \text{when} \quad t \geq M.$$

Then $L\{f(t)\} = F(s)$ exists for $s > a$.

Proof: The Laplace transform of $f$ is

$$L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt.$$
\[
\begin{align*}
&= \int_0^M f(t) e^{st} dt + \int_M^\infty f(t) e^{st} dt \\
\text{(now,)} \quad &\int_0^M f(t) e^{st} dt < \infty \text{ by (i)} \\
\text{and} \quad &\left| \int_M^\infty f(t) e^{st} dt \right| \leq \int_M^\infty |f(t)| e^{st} dt \\
&\leq \int_M^\infty Ke^{at} e^{st} dt \quad \text{by (ii)} \\
&= \int_M^\infty Ke^{(a-s) t} dt \\
&= \left. \frac{Ke^{(a-s) t}}{a-s} \right|_M^\infty \\
&= \lim_{t \to \infty} \frac{Ke^{(a-s) t}}{a-s} - \frac{Ke^{(a-s) M}}{a-s} \\
&= \frac{Ke^{(a-s) M}}{s-a} + \lim_{t \to \infty} \frac{Ke^{(a-s) t}}{a-s}
\end{align*}
\]
\[
L \{ f(t) \} = \mathcal{L} \{ f(t) \} = \begin{cases} 
\frac{c}{s} & \text{when } a < s \text{.} 
\end{cases}
\]

Some Laplace Transforms

1. \( f(t) = c \), \( c \) is some constant

\[
L \{ f(t) \} = \int_{0}^{\infty} f(t) e^{-st} \, dt
\]

\[
= \int_{0}^{\infty} ce^{-st} \, dt
\]

\[
= \left. \frac{ce^{-st}}{-s} \right|_{0}^{\infty}
\]

\[
= 0 - \frac{c}{-s}
\]

\[
= \frac{c}{s}
\]
2. \[ f(t) = e^{at} \]
\[
L\{f(t)\} = \int_0^\infty e^{at} e^{-st} dt
\]
\[
= \int_0^\infty e^{(a-s)t} dt
\]
\[
= \left. \frac{e^{(a-s)t}}{a-s} \right|_0^\infty
\]
\[
= 0 - \frac{1}{a-s}
\]
\[
= \frac{1}{s-a} \quad \text{when } a < S
\]

3. \[ f(t) = \sin(lat) \]
\[
L\{f(t)\} = \int_0^\infty \sin(lat) e^{-st} dt
\]
\[
= \left. \frac{e^{-st}\sin(lat)}{-s} \right|_0^\infty - \int_0^\infty \frac{\cos(\omega t)}{s} e^{-st} dt
\]
\[
u = \sin(lat) \quad du = e^{-st} dt
\]
\[
\begin{align*}
\frac{du}{dt} &= acos(lat) dt \quad \Rightarrow \quad \frac{du}{acos(lat) dt} = e^{-st} dt
\end{align*}
\]
\[
u = \frac{e^{-st}}{-s}
\]
\[ \begin{align*}
= 0 - 0 + \int_{0}^{\infty} \frac{a}{s} e^{-st} \cos(at) \, dt \\
= \frac{a}{s} \int_{0}^{\infty} e^{-st} \cos(at) \, dt \\
& \quad \text{if } u = \cos(at) \text{ then } dv = e^{-st} dt \\
& \quad \text{du} = -a \sin(at) \, dt \\
& \quad \text{v} = \frac{e^{-st}}{-s} \\
= \frac{a}{s} \left( \frac{e^{-st} \cos(at)}{-s} \bigg|_{0}^{\infty} - \int_{0}^{\infty} \frac{a}{s} e^{-st} \sin(at) \, dt \right) \\
= \frac{a}{s} \left( 0 - \frac{1}{-s} - a \int_{0}^{\infty} e^{-st} \sin(at) \, dt \right)
\end{align*} \]

Now, \( F(s) = \mathcal{L}\{f(t)\} \) so

\[ F(s) = \frac{a}{s} \left( \frac{1}{s} - \frac{a}{s} F(s) \right) \]

\[ F(s) + \frac{a^2}{s^2} F(s) = \frac{a}{s^2} \]

\[ F(s) = \frac{a}{s^2} \cdot \frac{1}{1 + \frac{a^2}{s^2}} \]
\[ = \frac{a}{s^2 + a^2} \]

Thus,
\[ \mathcal{L} \{ \sin(at) \} = \frac{a}{s^2 + a^2} \]

4. \( f(t) = \cos(at) \) (Similar process as 3.)
\[ \mathcal{L} \{ f(t) \} = \frac{s}{s^2 + a^2} \]

**Remark** Laplace Transform is a linear operator

\[ \mathcal{L} \{ c_1 f_1(t) + c_2 f_2(t) \} = \int_0^\infty (c_1 f_1(t) + c_2 f_2(t)) e^{-st} dt \]
\[ = \int_0^\infty c_1 f_1(t) e^{-st} dt + \int_0^\infty c_2 f_2(t) e^{-st} dt \]
\[ = c_1 \int_0^\infty f_1(t) e^{-st} dt + c_2 \int_0^\infty f_2(t) dt \]
\[ = C_1 \mathcal{L}\{f_1(t)\} + C_2 \mathcal{L}\{f_2(t)\} = C_1 F_1(s) + C_2 F_2(s) \]

**Ex** Find the Laplace transform of

\[ f(t) = 5e^{-2t} - 3\sin(4t) \]

\[ \mathcal{L}\{f(t)\} = \mathcal{L}\{5e^{-2t} - 3\sin(4t)\} \]

\[ = 5 \mathcal{L}\{e^{-2t}\} - 3 \mathcal{L}\{\sin(4t)\} \]

\[ = 5 \cdot \frac{1}{s+2} - 3 \cdot \frac{4}{s^2 + 16} \]

\[ = \frac{5}{s+2} - \frac{12}{s^2 + 16} \]

§6.2 Solution of Initial Value Problems

We will now see how the Laplace Transform can be used to solve initial value problems.
Thus, suppose \( f \) is continuous and \( f' \) is piecewise continuous on \( 0 \leq t \leq b \). Suppose that there are constants \( K, a, \) and \( M \) such that \( |f(t)| \leq Ke^{at} \) for \( t \geq M \). Then \( \int_{0}^{b} f'(t) \, dt \) exists and

\[
\int_{0}^{b} f'(t) \, dt = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f'(t) \, e^{-st} \, dt.
\]

Proof: Consider the integral

\[
\int_{0}^{b} e^{-st} f'(t) \, dt = \int_{t_0}^{t_1} f'(t) e^{-st} \, dt + \int_{t_1}^{t_2} f'(t) e^{-st} \, dt + \cdots + \int_{t_{n-1}}^{t_n} f'(t) e^{-st} \, dt
\]

\[
= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f'(t) e^{-st} \, dt.
\]

Since \( f'(t) \) is piecewise continuous. Now,
\[ \begin{align*}
\sum_{k=0}^{n-1} & \int_{t_k}^{t_{k+1}} f'(t) e^{-st} \, dt \\
& \quad - u = e^{-st} \quad dv = f(t) \, dt \\
& \quad du = -se^{-st} \, dt \quad v = f(t) \\
& = \sum_{k=0}^{n-1} f(t_k) e^{-st_0} + \int_{t_k}^{t_{k+1}} se^{-st} f(t) \, dt \\
& = f(t_1) e^{-st_1} - f(t_0) e^{-st_0} + f(t_2) e^{-st_2} - f(t_1) e^{-st_1} \\
& \quad + \cdots + f(t_n) e^{-st_n} - f(t_{n-1}) e^{-st_{n-1}} \\
& \quad + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} se^{-st} f(t) \, dt \\
& = f(t_n) e^{-st_n} - f(t_0) e^{-st_0} + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} se^{-st} f(t) \, dt \\
& = f(b) e^{-sb} - f(0) + s \int_0^b e^{-st} f(t) \, dt \\
\rightarrow & \quad s \int_0^\infty e^{-st} f(t) \, dt - f(0) \\
\text{as } b & \rightarrow \infty.
\end{align*} \]

Therefore,
\[ L\{f(t)\} = sL\{f(t)\} - f(0). \]
Remark: We can now use this result to find \( L\{f''(t)\} \).

\[
L\{f''(t)\} = S L\{f'(t)\} - f'(0)
= S(S L\{f(t)\} - f(0)) - f'(0)
= S^2 L\{f(t)\} - S f(0) - f'(0).
\]

For \( L\{f^{(n)}(t)\} \), we have the following corollary.

Cor: \( L\{f^{(n)}(t)\} = S^n L\{f(t)\} - S f^{(n-2)}(0) - f^{(n-1)}(0) \).

Ex: \( y'' - y' - 2y = 0 \), \( y(0) = 1 \), \( y'(0) = 0 \)

First, we transform to the \( S \)-domain.
Let $Y(s) = \mathcal{L}\{y(t)\}$. Then

\[ \mathcal{L}\{y''\} - y' - 2y = \mathcal{L}\{0\} \]

\[ \mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0 \]

Now,

\[ \mathcal{L}\{y''\} = s^2 \mathcal{L}\{y\} - sy(0) - y'(0) \]

\[ = s^2 Y(s) - s - 1 - 0 \]

\[ = s^2 Y(s) - s \]

and

\[ \mathcal{L}\{y'\} = s \mathcal{L}\{y\} - y(0) \]

\[ = sY(s) - 1 \]

Then, we have

\[ s^2 Y(s) - s - (sY(s) - 1) - 2Y(s) = 0 \]
Now, we solve for $Y(s)$,

$$S^2 Y(s) - S - SY(s) + 1 - 2Y(s) = 0$$

$$Y(s) \left( S^2 - S - 2 \right) = S - 1$$

$$Y(s) = \frac{S - 1}{S^2 - S - 2}$$

We have found the solution in the $S$-domain. The final step is to go back to the $t$-domain.

Remark: Notice that our differential equation became an algebraic equation. This vastly simplifies the problem, but we are now left with returning to the $t$-domain.
Inverting the Laplace Transform

We will not explicitly use the inverse Laplace Transform, but rather compare the solution \( Y(s) \) to known transformed functions to infer the inversion. Again, the key idea is linearity.

Suppose

\[
F(s) = F_1(s) + F_2(s) + \cdots + F_n(s)
\]

and

\[
f(t) = \mathcal{L}^{-1} \{ F_1(s) \}, \quad \cdots, \quad f_n(t) = \mathcal{L}^{-1} \{ F_n(s) \}
\]

then

\[
f(t) = \mathcal{L}^{-1} \{ F(s) \} = \mathcal{L}^{-1} \{ F_1(s) \} + \cdots + \mathcal{L}^{-1} \{ F_n(s) \}.
\]