Inverting the Laplace transform

Ex. We found

\[ Y(s) = \frac{S-1}{s^2 s-2} \]

so let's go back to the \( t \)-domain.
We know

- \( \mathcal{L}\{e^t\} = \frac{1}{s-1} \), \( c \) is a constant
- \( \mathcal{L}\{e^{at}\} = \frac{1}{s-a} \)
- \( \mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2} \)
- \( \mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2} \)

Rewriting \( Y(s) \) using partial fractions, we have

\[ Y(s) = \frac{s-1}{(s-2)(s+1)} \]
\[
\gamma(s) = \frac{A}{s-2} + \frac{B}{s+1}
\]

So,
\[
A(s+1) + B(s-2) = s-1
\]
\[
S(A + B) + A - 2B = s - 1
\]
\[
A + B = 1, \quad A - 2B = -1
\]
\[
A = 1 - B \Rightarrow 1 - B - 2B = -1
\]
\[
-3B = -2
\]
\[
B = \frac{2}{3}
\]
\[
\Rightarrow A = \frac{1}{3}
\]

Therefore,
\[
\gamma(s) = \frac{1}{3} \frac{1}{s-2} + \frac{2}{3} \frac{1}{s+1}
\]

Now,
\[
\mathcal{L}^{-1}\{\gamma(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{3} \frac{1}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{3} \frac{1}{s+1}\right\}
\]
Thus,

\[ y(t) = \frac{1}{3} e^{2t} + \frac{2}{3} e^{t} \]
Chapter 7: Systems of First-Order Linear Equations

§7.1 Introduction

We will consider systems of ODEs and denote the independent variable by $t$, and the dependent variables as $x_1, x_2, \ldots, x_n$ which are functions of $t$.

Derivatives will be denoted as $\frac{dx_i}{dt}$ or $x_i'$ (\(\ddot{x}_i\) is also common).

Why study systems of ODEs?

Higher order equations can be reformulated as a system of first order ODEs.

Ex: Consider the general equation for a spring-mass system
\[ m u'' + \gamma u' + ku = F(t) \].

Let \( x_1 = u \) and \( x_2 = u' \). Then
\[
\begin{align*}
x_1' &= u' = x_2 \\
x_2' &= u'' = \frac{1}{m} (F(t) - \gamma u' - ku) = \frac{1}{m} (F(t) - \gamma x_2 - kx_1)
\end{align*}
\]
and so in matrix form, we have the system
\[
\begin{bmatrix}
x_1' \\
x_2'
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-\frac{k}{m} & -\frac{\gamma}{m}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
\frac{F(t)}{m}
\end{bmatrix}.
\]

For an arbitrary \( n \)th order equation
\[ y^{(n)} = F(t, y, y', \ldots, y^{(n-1)}) \]
we let
\[
\begin{align*}
x_1 &= y \\
x_2 &= y' \\
x_3 &= y'' \\
&\vdots
\end{align*}
\]
\[ \Rightarrow \]
\[
\begin{align*}
x_1' &= y' = x_2 \\
x_2' &= y'' = x_3 \\
&\vdots
\end{align*}
\]
\[ x_n = y^{(n)} \quad \quad x_n^1 = y^{(n)} = F(t, x_1, x_2, \cdots, x_n) \]

More generally, the system of \( n \) first order ODEs is:

\[ x_1^1 = F_1(t, x_1, x_2, \cdots, x_n) \]
\[ x_2^1 = F_2(t, x_1, x_2, \cdots, x_n) \]
\[ \vdots \]
\[ x_n^1 = F_n(t, x_1, x_2, \cdots, x_n) \]

and the solution of such a system consists of \( n \) functions:

\[ x_1 = \varphi_1(t) \]
\[ x_2 = \varphi_2(t) \]
\[ \vdots \]
\[ x_n = \varphi_n(t) \]

We may also prescribe initial conditions:

\[ x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0, \quad \cdots, \quad x_n(t_0) = x_n^0. \]
We can think of the solution as a set of parametric equations in $n$-dimensional space.

Thus (Existence and Uniqueness)

let each of the $n$ functions $F_1, \ldots, F_n$

and the $n^2$ partial derivatives $\frac{\partial F_i}{\partial x_1}, \ldots, \frac{\partial F_i}{\partial x_n}, \ldots, \frac{\partial F_n}{\partial x_1}, \ldots, \frac{\partial F_n}{\partial x_n}$

be continuous in a region $R$
of $t \times x_1, x_2, \ldots, x_n$-space defined by $2 < t < 3$, $\alpha_1 < x_1 < \beta_1$, $\ldots$, $\alpha_n < x_n < \beta_n$ and let

$(t_0, x_1^0, x_2^0, \ldots, x_n^0) \in R$. Then there is an interval $|t_0 - t| < h$ in which there exists a unique solution $\bar{x} = \bar{\phi}$ of the system of ODEs that satisfies the IC,
Linear and Nonlinear Systems

If $F_1, \ldots, F_n$ are linear functions of $x_1, \ldots, x_n$ then the system is called linear. Otherwise, it is nonlinear.

General system of $n$ first-order linear ODEs

\[
\begin{align*}
    x'_1 & = P_{11}(t) \cdot x_1 + \cdots + P_{1n}(t) \cdot x_n + g_1(t) \\
    x'_2 & = P_{21}(t) \cdot x_1 + \cdots + P_{2n}(t) \cdot x_n + g_2(t) \\
    & \quad \vdots \\
    x'_n & = P_{n1}(t) \cdot x_1 + \cdots + P_{nn}(t) \cdot x_n + g_n(t)
\end{align*}
\]

If $g_i(t) = 0$ for $t \in I$, $i = 1, \ldots, n$, the system is called homogeneous. Otherwise, it is called nonhomogeneous.
(Existence and Uniqueness for Linear Systems)

If $p_{ij}(t)$ and $q_{i}(t)$, $i,j=1,...,n$ are continuous on $I : a < t < b$, then there exists a unique solution $\vec{x} = \vec{\phi}$ that satisfies the ICs. The solution exists throughout the interval $I$.

We will spend the remainder of our time studying linear systems of first-order ODEs.
7.2 Matrices

The \( m \times n \) matrix with entries \( a_{ij} \) is

\[
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]

The transpose of \( A \) is denoted \( A^T \) and if \( A = (a_{ij}) \), \( A^T = (a_{ji}) \).

The conjugate of \( A \) is denoted \( \bar{A} \) and \( \bar{A} = (\bar{a_{ij}}) \).

The adjoint of \( A \) is \( \bar{A}^T \) or \( A^* = (\bar{a_{ji}}) \).

We will focus on square matrices \((m=n)\) and (column) vectors.
Properties of Matrices

1. Equality: $A$ and $B$ are equal if $a_{ij} = b_{ij}$ for each $i$ and $j$

2. Zero: $\vec{0}$ will denote the matrix or vector whose entries are all zero.

3. Addition: $A + B = (a_{ij}) + (b_{ij})$

4. Scalar Multiplication: $\lambda A = \lambda (a_{ij}) = (\lambda a_{ij})$

5. Matrix Multiplication: $C = AB$

$$C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Note: Matrix multiplication is not commutative $AB \neq BA$
6. Multiplication of vectors (dot product)
\[ \mathbf{x}^T \mathbf{y} = \sum_{i=1}^{n} x_i y_i \quad \text{This produces a scalar} \]

Inner product \( \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i \)

Length/magnitude of \( \mathbf{x} \) is \( \|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \)

- if \( \langle \mathbf{x}, \mathbf{x} \rangle = 0 \), must have that \( \mathbf{x} = \mathbf{0} \).

If \( \langle \mathbf{x}, \mathbf{y} \rangle = 0 \), then \( \mathbf{x} \) and \( \mathbf{y} \) are orthogonal.

7. Identity matrix \( \mathbf{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \)

\( \mathbf{A} \mathbf{I} = \mathbf{I} \mathbf{A} = \mathbf{A} \)

8. Inverse and Determinant

- \( \mathbf{A} \) is called nonsingular/invertible if there is a matrix \( \mathbf{B} \) so that \( \mathbf{AB} = \mathbf{I} \) and \( \mathbf{BA} = \mathbf{I} \). We write \( \mathbf{B} = \mathbf{A}^{-1} \).
If $A$ has no inverse, we call $A$ singular or noninvertible.

$A$ is nonsingular $\iff \det A \neq 0$.

**Matrix Functions**

We will consider vectors and matrices whose elements are functions.

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix}$$

$A(t)$ is called continuous if each element of $A$ is continuous.

Similarly,

$$\frac{dA}{dt} = \begin{pmatrix} \frac{da_{ij}}{dt} \end{pmatrix}$$
\[
\int_{a}^{b} A(t) \, dt = \left( \int_{a}^{b} a_{ij} \, dt \right)
\]

\[
F_{x} = A(t) = \begin{pmatrix} \sin t & t \\ 0 & \cos t \end{pmatrix}
\]

\[
A'(t) = \begin{pmatrix} \cos t & 1 \\ 0 & -\sin t \end{pmatrix}
\]

\[
\left[ \int_{0}^{\pi} A(t) \, dt \right] = \left[ \begin{pmatrix} -\cos t & \frac{t^2}{2} \\ t & -\sin t \end{pmatrix} \right]_{0}^{\pi}
\]

\[
= \begin{pmatrix} 2 & \frac{\pi^2}{2} \\ \pi & 0 \end{pmatrix}
\]

Many calculus rules hold:

\[
\frac{d}{dt} (CA) = C \frac{dA}{dt}
\]

where \( C \) is a constant matrix.
\[ \frac{d}{dt} (A+B) = \frac{dA}{dt} + \frac{dB}{dt} \]

\[ \frac{d}{dt} (AB) = A \frac{dB}{dt} + \frac{dA}{dt} B \]

7.3 Systems of Linear Algebraic Equations:

Linear Independence, Eigenvalues and Eigenvectors

Systems of Linear Algebraic Equations

\[ a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1, \]

\[ \vdots \]

\[ a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n = b_n \]

or

\[ A \vec{x} = \vec{b} \]
If \( \bar{b} = 0 \), the system is called homogeneous. Otherwise it is nonhomogeneous.

If \( A \) is nonsingular, then \( \bar{x} = A^{-1} \bar{b} \), and the homogeneous problem only has the solution \( \bar{x} = \bar{0} \).

If \( A \) is singular, then \( A\bar{x} = \bar{b} \) has no solution or a nonunique solution. The homogeneous problem \( A\bar{x} = \bar{0} \) has infinitely many solutions.

When does \( A\bar{x} = \bar{b} \) have a solution?

We must have that \( \langle \bar{b}, \bar{y} \rangle = 0 \) for all \( \bar{y} \) satisfying \( A^*\bar{y} = \bar{0} \).
Thus there are infinitely many solutions of the form
\[ \mathbf{x} = \mathbf{x}^{(0)} + \mathbf{\xi} \]
where
\[ A\mathbf{x}^{(0)} = \mathbf{b} \quad \text{and} \quad A\mathbf{\xi} = \mathbf{0}. \]

---

**Linear Dependence and Independence**

\( \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)} \) are linearly dependent if there are scalars \( c_1, \ldots, c_k \) (with at least one non-zero) such that
\[ c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} + \cdots + c_k \mathbf{x}^{(k)} = \mathbf{0}. \]

If the only solution is \( c_1 = c_2 = \cdots = c_k = 0 \) then \( \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)} \) are linearly independent.
Eigenvalues and Eigenvectors

We may consider $A\vec{x} = \vec{y}$ as a linear transformation that maps $\vec{x}$ to $\vec{y}$. We are interested in finding vectors that are mapped to a scalar multiple of itself. I.e. $\vec{y} = \lambda \vec{x}$. Then

$$A\vec{x} = \lambda \vec{x}$$

$$A\vec{x} - \lambda \vec{x} = 0$$

$$(A - \lambda I)\vec{x} = 0$$

This equation has nonzero solutions if and only if

$$\det(A - \lambda I) = 0 \quad \text{(characteristic equation)}$$

$\lambda$ is called an eigenvalue

$\vec{x}$ is called an eigenvector
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