§ 2.8 The Existence and Uniqueness Theorem

**Theorem Existence and Uniqueness**

If $f$ and $\frac{df}{dy}$ are continuous in a rectangle $R: |t| \leq a$, $|y| \leq b$, then there is some interval $|t| \leq h \leq a$ in which there exists a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), \quad y(0) = 0.$$ 

**Note** If we have the IVP $y' = f(t, y)$, $y(t_0) = y_0$, we can make a change of variables so that the initial point $(t_0, y_0)$ is the origin.
Now, we will examine elements of the existence/uniqueness theorem.

Suppose \( y = \phi(t) \) satisfies the IVP. Then \( f(t, y) = f(t, \phi(t)) \), so \( f \) only depends on \( t \). Now,

\[
y' = f(t, y) = f(t, \phi(t))
\]

\[
\int_{0}^{t} y' \, dt = \int_{0}^{t} f(s, \phi(s)) \, ds
\]

\[
y(t) - y(0) = \int_{0}^{t} f(s, \phi(s)) \, ds
\]

\[
y(t) = \int_{0}^{t} f(s, \phi(s)) \, ds.
\]

Thus expression for \( y(t) \) involves the
integral of an unknown function $\Phi$ and is called an integral equation. 

The integral equation and IVP are equivalent. 

Picard's iteration method (method of successive approximations) 

- Method for showing the integral equation has a unique solution

We generate a sequence of functions $\Phi_n(t)$ as follows:

1. Choose $\Phi_0$. Simplest choice is $\Phi_0(t) = 0$. 
2. We get $\Phi_n$ by using $\Phi_{n-1}$ in the integral equation.
\[ \varphi_i(t) = \int_0^t f(s, \varphi_0(s)) \, ds \]

3. We get \( \varphi_2 \) by using \( \varphi_1 \)
\[ \varphi_2(t) = \int_0^t f(s, \varphi_1(s)) \, ds \]
and so on. In general,
\[ \varphi_{n+1}(t) = \int_0^t f(s, \varphi_n(s)) \, ds \].

We now have a sequence of functions. Each function in the sequence satisfies the IC, but may not satisfy the DE. If at some point, say \( n = k \), we have \( \varphi_{k+1}(t) = \varphi_k(t) \), then \( \varphi_k \) is a solution of the integral
equation.

To prove the Existence/Uniqueness theorem we need to know:

1. Do all members of \( \{ f_n \} \) exist?
2. Does \( \{ f_n \} \) converge?
3. What are the properties of the limit function?
4. Is this the only solution?

1. Do all members of \( \{ f_n \} \) exist?

\( f \) and \( \frac{df}{dy} \) are continuous in the rectangle \( R: |x| \leq a, |y| \leq b \) so the
danger is that \( y = \Phi_k(t) \) may be outside of \( R \). We need to restrict \( t \) to a smaller interval than \( |t| \leq a \).

It is continuous on a closed bounded region so there is constant \( M > 0 \) such that
\[
|f(t, y)| \leq M \text{ for } (t, y) \in R.
\]

Now,
\[
\Phi'_{k+1} = f(t, \Phi_k) \leq M
\]
so the point \((t, \Phi_{k+1}(t)) \in R\) as long as \(|t| \leq \frac{b}{M}\).
Choose \( h = \min \{ a, \frac{b}{M} \} \).

2. Does \( \varphi_n \) converge?

We see that

\[
\varphi_n(t) = \varphi_1 + (\varphi_2 - \varphi_1) + \ldots + (\varphi_n - \varphi_{n-1})
\]
is the partial sum of
\[ \Phi_1 + \sum_{k=1}^{\infty} (\Phi_{k+1} - \Phi_k). \]

Therefore, if the series converges, then the sequence \( \{\Phi_n\} \) converges and we let
\[ \Phi(k) = \lim_{n \to \infty} \Phi_n(k). \]

3. What are the properties of the limit function?

We want to know if \( \Phi \) is continuous. If each \( \Phi_n \) is continuous and \( \{\Phi_n\} \) "converges uniformly" then \( \Phi \) is continuous.
\[ \varphi_{n+1}(t) = \int_0^t f(s, \varphi_n(s)) \, ds \]

\[ \varphi(t) = \lim_{n \to \infty} \int_0^t f(s, \varphi_n(s)) \, ds \]

\[ = \int_0^t \lim_{n \to \infty} f(s, \varphi_n(s)) \, ds \]

\[ = \int_0^t f(s, \varphi(s)) \, ds \]

Since \( \varphi_n \) converges uniformly.
Since \( f \) is continuous.

4. Is this the only solution?

Suppose \( y_1 = \varphi(t) \) and \( y_2 = \psi(t) \) are both solutions. Then

\[ |y_1 - y_2| = |\varphi(t) - \psi(t)| \]
\[
\begin{align*}
&= \left| \int_0^t f(s, \varphi(s)) \, ds - \int_0^t f(s, \psi(s)) \, ds \right| \\
&= \left| \int_0^t f(s, \varphi(s)) - f(s, \psi(s)) \, ds \right| \\
&\leq \int_0^t |f(s, \varphi(s)) - f(s, \psi(s))| \, ds \\
&\leq \int_0^t \ell |\varphi(s) - \psi(s)| \, ds
\end{align*}
\]

for some constant \( \ell \) (Lipschitz constant)

hence

\[
A(t) = \int_0^t |\varphi(s) - \psi(s)| \, ds
\]

Then

\[
A(0) = 0
\]

\[
A(t) \geq 0 \quad \text{for} \quad t \geq 0
\]

and

\[
A'(t) = |\varphi(t) - \psi(t)|
\]
Now,

$$|\phi(t) - \psi(t)| \leq L \int_0^t |\phi(s) - \psi(s)| ds$$

$$A'(t) \leq LA(t)$$

$$A'(t) - LA(t) \leq 0.$$  

Multiply by $e^{-Lt}$, we get

$$\frac{d}{dt} (e^{-Lt} A(t)) \leq 0$$

so

$$e^{-Lt} A(t) \leq 0.$$  

Hence

$$A(t) \leq 0.$$  

We now have that

$$A(0) = 0$$  and  $$A(t) \geq 0$$  for $t \geq 0$.  

AND

$A(t) \leq 0 \text{ for } t \geq 0$.

So $A(t) = 0 \text{ for } t \geq 0$.

Hence, $A'(t) = 0$

So

$|\psi(t) - \psi(t)| = 0$

i.e. $\psi(t) = \psi(t) \text{ for } t \geq 0$. 
§ 2.9 First-Order Difference Equations

So far, we have considered continuous models but now we explore discrete models which lead to difference equations.

A first-order difference equation is of the form

\[ y_{n+1} = f(n, y_n), \quad n = 0, 1, 2, \ldots \]

It is first-order since \( y_{n+1} \) only depends on \( y_n \).

The difference eqn is linear if \( f \) is a linear function of \( y_n \), otherwise it is nonlinear.

A solution is a sequence of numbers \( y_0, y_1, \ldots \) that satisfy the equation for
each $n$.

The initial condition $y_0 = a$, specifies the first term of the solution sequence.

**Equilibrium Solutions**

Suppose $y_{n+1} = f(y_n)$, $n = 0, 1, 2, \ldots$

where $f$ depends only on $y_n$. Then

\[ y_1 = f(y_0) \]

\[ y_2 = f(y_1) = f(f(y_0)) \]

\[ y_3 = f(y_2) = f(f(f(y_0))) = f^3(y_0) \]

and so

\[ y_n = f^n(y_0) \]
We may want to know what happens to \( y_n \) as \( n \to \infty \). If \( y_n \) has the same value for all \( n \), \( y_n \) is called an equilibrium solution. Such a solution satisfies
\[
y_n = f(y_n).
\]

**Linear Equations**

Consider the difference equation
\[
y_{n+1} = gy_n + b, \quad n = 0, 1, 2, \ldots
\]

First, let's solve the equation in terms of the initial value \( y_0 \). We have
\[ y_1 = s y_0 + b_0 \]
\[ y_2 = s y_1 + b_1 = s(s y_0 + b_0) + b_1 = s^2 y_0 + s b_0 + b_1, \]
\[ y_3 = s y_2 + b_2 \]
\[ = s(s^2 y_0 + s b_0 + b_1) + b_2 = s^3 y_0 + s^2 b_0 + s b_1 + b_2 \]

and so
\[ y_n = s^n y_0 + s^{n-1} b_0 + \ldots + s b_{n-2} + b_{n-1} \]
\[ = s^n y_0 + \sum_{j=0}^{n-1} s^{n-1-j} b_j. \]

Suppose \( b_n = b \neq 0 \) for all \( n \), then
\[ y_n = s^n y_0 + b \sum_{j=0}^{n-1} s^{n-1-j} \]
\[ = s^n y_0 + b \frac{1 - s^n}{1-s} \]
\[ = s^n (y_0 - \frac{b}{1-s}) + \frac{b}{1-s} \]

Now, let's determine the long-time behavior of \( y_n \).

1. \( |s| < 1 \) \( \implies \) \( y_n \to \frac{b}{1-s} \)

2. \( |s| > 1 \)
   
   a. \( y_0 = \frac{b}{1-s} \) \( \implies \) \( y_n = \frac{b}{1-s} \) for all \( n \)
   
   b. \( y_0 \neq \frac{b}{1-s} \) \( \implies \) no limit

3. \( s = -1 \) \( \implies \) no limit
4. $s = 1$, we need to go back to the difference equation

$$y_{n+1} = sy_n + b = y_n + b$$

$$\implies y_n \to \infty \text{ as } n \to \infty$$

**Nonlinear Equations**

**Logistic difference equation**

$$y_{n+1} = sy_n \left(1 - \frac{y_n}{K}\right)$$

or

$$u_{n+1} = su_n \left(1 - u_n\right)$$

where $u_n = \frac{y_n}{K}$. 
Let's find equilibrium solutions

$$u_n = gu_n (1-u_n)$$

$$gu_n^2 + u_n (1-g) = 0$$

$$u_n (gu_n + 1-g) = 0$$

$$u_n = 0 \quad u_n = \frac{g-1}{g}$$

Cobweb/Stairstep diagram