Ch. 3  Second-Order Linear Differential Equations

§3.1  Homogeneous DEs with constant coefficients

Consider the 2nd-order DE of the form

$$\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt})$$

which is linear if

$$f(t, y, \frac{dy}{dt}) = g(t) - p(t) \frac{dy}{dt} - q(t)y$$

where $g, p, \text{ and } q$ are given functions of $t$. Our general 2nd-order linear DE is

$$y'' + p(t)y' + q(t)y = g(t).$$

The initial value problem requires us to specify

$$y(t_0) = y_0 \quad \text{and} \quad y'(t_0) = y'_0.$$
We now specify an initial point \((t_0, y_0)\) and an initial slope \(y_0'\).

The second order equation is called \textbf{homogeneous} if \(g(t) = 0\) for all \(t\). Otherwise, it is called \textbf{nonhomogeneous}.

\textbf{Note:} Solving the nonhomogeneous problem requires solving the homogeneous problem first, so we examine the homogeneous case first.

Consider
\[
ay'' + by' + cy = 0
\]
where \(a\), \(b\) and \(c\) are given constants. We have already seen how to solve this DE!
Idea 1: Suppose \( y = e^{rt} \) is a solution to the DE. Then

\[
ay'' + by' + cy = 0
\]

\[
a(e^{rt})'' + b(e^{rt})' + ce^{rt} = 0
\]

\[
ar^2 e^{rt} + be^{rt} + ce^{rt} = 0
\]

\[
e^{rt}(ar^2 + br + c) = 0
\]

Now, \( e^{rt} \neq 0 \), so

\[
ar^2 + br + c = 0.
\]

This is called the characteristic equation for the DE. The characteristic equation has two roots \( r_1 \) and \( r_2 \) which may be real and distinct, real and
repeated, or complex conjugates depending on \(a, b,\) and \(c.\)

For now, let us assume \(r_1\) and \(r_2\) are distinct real roots. Then
\[
y_1 = e^{r_1 t} \quad \text{and} \quad y_2 = e^{r_2 t}
\]
both solve the DE.

\underline{Idea 2:} If \(y_1\) and \(y_2\) are solutions of the DE, then
\[
y = c_1 y_1 + c_2 y_2
\]
solves the DE.

\[
ay'' + by' + cy = a (c_1 y_1 + c_2 y_2)'' + b (c_1 y_1 + c_2 y_2)' + c (c_1 y_1 + c_2 y_2)
= ac_1 y_1'' + bc_1 y_1' + c c_1 y_1,
\]
\[
+ ac_2 y_2'' + bc_2 y_2' + c c_2 y_2
\]
\[ y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \]

\[ = 0 \]

Since $r_1$ and $r_2$ are roots of $ax^2 + bx + c$.

The Initial Value Problem

We have our solution

\[ y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \]

and we can get a particular solution using the initial values

\[ y(t_0) = y_0 \quad \text{and} \quad y'(t_0) = y'_0 \]

to determine $c_1$ and $c_2$.

\[ y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \]

is called the **general solution**.
Ex Solve the IVP

\[ 4y'' - 8y' + 3y = 0 \, , \, y(0) = 2, \, y'(0) = \frac{1}{2} \]

First, the characteristic equation is

\[ 4r^2 - 8r + 3 = 0 \]
\[ (2r - 1)(2r - 3) = 0 \]
\[ r_1 = \frac{3}{2} \, \quad r_2 = \frac{1}{2} \]

Therefore, our general solution is

\[ y = c_1 e^{\frac{3}{2}t} + c_2 e^{\frac{1}{2}t} \]

Now, we determine \( c_1 \) and \( c_2 \) using the ICs

\[ 2 = y(0) = c_1 + c_2 \]

so

\[ c_1 = 2 - c_2 \]
Now,
\[ y = (2 - c_2) e^{\frac{3}{2}t} + c_2 e^{\frac{1}{2}t} \]
and
\[ y' = \frac{3}{2} (2 - c_2) e^{\frac{3}{2}t} + \frac{1}{2} c_2 e^{\frac{1}{2}t} \]
We have
\[ \frac{1}{2} = y'(0) = \frac{3}{2} (2 - c_2) + \frac{1}{2} c_2 \]
\[ 1 = 3 (2 - c_2) + c_2 \]
\[ -5 = -2 c_2 \]
\[ c_2 = \frac{5}{2} \]
Then
\[ c_1 = 2 - c_2 \]
\[ = 2 - \frac{5}{2} \]
\[ = -\frac{1}{2} \].
Thus,
\[ y = -\frac{1}{2} e^{\frac{3}{2}t} + \frac{5}{2} e^{\frac{1}{2}t} \]

**Long-time behavior**

Our general solution is
\[ y = c_1 e^{rt} + c_2 e^{nt} \]

What happens as \( t \to \infty \)?
§3.2 Solutions of Homogeneous Equations; Wronskian

Let $p$ and $q$ be continuous functions on an open interval $I$. The differential operator $L$ is defined by

$$L[\phi] = \phi'' + p(t)\phi' + q(t)\phi$$

Where $\phi$ is some function.

Note: $L$ operates on $\phi$, i.e. $\phi$ is the input to $L$.

The value of $L$ at a point $t$ is

$$L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t)$$

We will examine the second-order linear DE $L[\phi](t) = 0$ with ICs $y(t_0) = y_0$, $y'(t_0) = y'_0$.

We want to know whether the IVP
has a solution, more than one solution, and what is the structure and form of solutions.

**Theorem (Existence and Uniqueness)**

Let $p, q,$ and $g$ be continuous in an open interval $I$ containing $t_0$. Then the IVP

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0$$

has a unique solution $y = g(t)$ in the interval $I$.

**Theorem (Principle of Superposition)**

If $y_1$ and $y_2$ are solutions to the DE

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

then the linear combination $c_1y_1 + c_2y_2$
is also a solution for any \( c_1 \) and \( c_2 \).

**Proof** Follows from the linearity of \( \mathcal{L} \).

**Q:** When can \( c_1 \) and \( c_2 \) be chosen to satisfy the ICs \( y(0) = y_0 \) and \( y'(0) = y_0' \)?

We need

\[
\begin{align*}
    c_1 y_1(0) + c_2 y_2(0) &= y_0 \\
    c_1 y_1'(0) + c_2 y_2'(0) &= y_0'
\end{align*}
\]

in matrix form

\[
\begin{bmatrix}
    y_1(0) & y_2(0) \\
    y_1'(0) & y_2'(0)
\end{bmatrix}
\begin{bmatrix}
    c_1 \\
    c_2
\end{bmatrix}
=
\begin{bmatrix}
    y_0 \\
    y_0'
\end{bmatrix}
\]
When does this system have a unique solution?

\[ W(t_0) = \det \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}. \]

\( W(t_0) \neq 0 \) guarantees a unique solution \( \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \).

\( W \) is called the Wronskian of the solutions \( y_1 \) and \( y_2 \).

Thus, suppose \( y_1 \) and \( y_2 \) are two solutions of

\[ L[y] = y'' + p(t)y' + q(t)y = 0. \]

It is possible to choose \( C_1 \) and
\[ c_2 \text{ so that } \]
\[ y = c_1 y_1(t) + c_2 y_2(t) \]
Satisfies the DE and ICs \( y(t_0) = y_0 \), \( y'(t_0) = y'_0 \) if and only if the Wronskian
\[ W[y_1, y_2] = y_1 y_2' - y_1' y_2 \]
is not zero at \( t_0 \).

**Ex** Consider the DE
\[ y'' + 5y' + 6y = 0 \]
Find the Wronskian of \( y_1 \) and \( y_2 \).
The characteristic equation is
\[ r^2 + 5r + 6 = 0 \]
\[ (r + 3)(r + 2) = 0 \]
\[ r = -3 \quad r = -2 \]

So \[ y_1 = e^{-2t} \] and \[ y_2 = e^{-3t} \].

Now,

\[
W[y_1, y_2] = \begin{vmatrix} -2t & -3t \\ e^t & e \\ -2e^{2t} & -3e^{3t} \end{vmatrix} = -3e^{5t} + 2e^{5t} = -e^{5t}
\]

We see that \( W \neq 0 \) for all \( t \), so any IC can be specified at any initial time \( t_0 \).

The expression

\[ y = c_1 y_1(t) + c_2 y_2(t) \]
is called the **general solution** of \( L[y] = 0 \). \( y_1 \) and \( y_2 \) are said to form a **fundamental set of solutions** if and only if their Wronskian is nonzero for all \( t \).

**Remark** Nonzero Wronskian \( \iff \) \( y_1 \) and \( y_2 \) are linearly independent

**Ex** Show that \( y_1 = t^{\frac{1}{2}} \) and \( y_2 = \frac{1}{t} \) form a fundamental set of solutions of

\[
2t^2 y'' + 3ty' - y = 0, \quad t > 0.
\]

1st we need to show \( y_1 \) and \( y_2 \) are solutions.

\[
2t^2 y_1'' + 3ty_1' - y_1 = 2t^2 \left( t^{\frac{1}{2}} \right)'' + 3t \left( t^{\frac{1}{2}} \right)' - t^{\frac{1}{2}}
\]

\[
= 2t^2 \left( \frac{1}{2} t^{-\frac{1}{2}} \right)' + 3t \frac{1}{2} t^{\frac{3}{2}} - t^{\frac{1}{2}}
\]

\[
= 2t^2 \frac{1}{2} t^{\frac{3}{2}} + \frac{3}{2} t^{\frac{1}{2}} - t^{\frac{1}{2}}
\]
\[ \frac{1}{2} t^2 + \frac{1}{2} t^2 \]
\[ = 0 \checkmark \]

so \( y_1 \) is a solution.

\[
2 t^2 y_2^{\prime\prime} + 3 t y_2^{\prime} - y_2 = 2 t^2 \left( \frac{1}{t^2} \right)^{\prime\prime} + 3 t \left( \frac{1}{t^2} \right)^{\prime} - \frac{1}{t} \\
= 2 t^2 \left( \frac{-1}{t^2} \right)^{\prime} + 3 t \frac{-1}{t^2} - \frac{1}{t} \\
= 2 t^2 \frac{2}{t^3} + \frac{-3}{t} - \frac{1}{t} \\
= \frac{2}{t} - \frac{3}{t} - \frac{1}{t} \\
= 0 \checkmark
\]

so \( y_2 \) is a solution.

Now, we calculate the Wronskian

\[
W = \begin{vmatrix}
\frac{1}{2} t^2 & \frac{1}{t} \\
\frac{1}{2} t^\frac{1}{2} & -\frac{1}{t^2}
\end{vmatrix}
\]
\[ t^\frac{1}{2}(-t^2) - \frac{1}{2} t^\frac{1}{2} t^{-1} \]
\[ = -t^\frac{3}{2} - \frac{1}{2} t^\frac{3}{2} \]
\[ = -\frac{3}{2} t^\frac{3}{2} \]

So \( \mathbf{W} \neq 0 \) for \( t > 0 \). Thus, \( y, \) and \( y_2 \) form a fundamental set of solutions and the general solution is
\[ y(t) = c_1 t^\frac{1}{2} + c_2 \frac{1}{t^\frac{1}{2}} . \]

Thus if \( y = u(t) + iv(t) \) is a complex-valued solution of \( L[\mathbf{y}] = 0 \), then its real part \( u \) and imaginary part \( v \) are also solutions.

Proof: Follows from linearity of \( L \) and
\[ L[y] = 0. \]

*Hint:* If \( a \cdot b = 0 \), then \( a = 0 \) and \( b = 0 \).

**Theorem (Abel's Theorem)**

If \( y_1 \) and \( y_2 \) are solutions of

\[ L[y] = y'' + p(t)y' + q(t)y = 0 \]

where \( p \) and \( q \) are continuous on \( I \),

then the Wronskian is

\[ W[y_1, y_2](t) = C \ e^{-\int p(t) \, dt} \]

where \( C \) is a constant depending on \( y_1 \) and \( y_2 \) but not on \( t \). \( W \) is either zero for all \( t \) in \( I \) or never zero in \( I \).