§ 3.5 Nonhomogeneous Equations: Method of Undetermined Coefficients

We now consider the nonhomogeneous second-order linear DE

$$\mathcal{L}[y] = y'' + p(t)y' + q(t)y = g(t) \quad (*)$$

When $g(t) = 0$, we call the resulting DE the homogeneous DE corresponding to $(*)$.

Thus if $Y_1$ and $Y_2$ are solutions of the nonhomogeneous linear DE $\mathcal{L}[y] = g(t)$, then $Y_1 - Y_2$ is a solution of the homogeneous DE $\mathcal{L}[y] = 0$. If $y_1$ and $y_2$ form a fundamental set of solutions to $\mathcal{L}[y] = 0$, then

$$Y_1 - Y_2 = c_1 y_1 + c_2 y_2.$$
Proof \( Y_1 \) and \( Y_2 \) satisfy \( \mathcal{L}\{Y_1\} = g(t) \) and \( \mathcal{L}\{Y_2\} = g(t) \). Then

\[
\mathcal{L}\{Y_1 - Y_2\} = \mathcal{L}\{Y_1\} - \mathcal{L}\{Y_2\} = g(t) - g(t) = 0
\]

so \( Y_1 - Y_2 \) is a solution of \( \mathcal{L}\{y\} = 0 \).

Since any solution of \( \mathcal{L}\{y\} = 0 \) is a linear combination of the fundamental solutions

\[
Y_1 - Y_2 = c_1 Y_1 + c_2 Y_2
\]

Therefore, the general solution of \( \mathcal{L}\{y\} = g(t) \) is of the form

\[
y = g(t) = c_1 Y_1 + c_2 Y_2 + Y
\]
where $y_1$ and $y_2$ form a fundamental set of solutions of $L[y] = 0$ and $Y$ is any solution of $L[y] = g(t)$.

Steps for solving $L[y] = g(t)$

1. Find general solution to homogeneous equation $L[y] = 0$
   
   \[ y_c = c_1y_1 + c_2y_2, \text{ called the complementary solution} \]

2. Find solution to $L[y] = g(t)$, $y_p$ called the particular solution.

3. The general is

   \[ y = y_c + y_p \]

How to find $y_p$?
Method of Undetermined Coefficients

Idea: Based on $g(t)$, we make an ansatz on the form of $y_p$ which will involve arbitrary constants.

- Find constants so that $L[y_p] = g(t)$.

Example: $y'' - 3y' + 4y = 3e^{2t}$. Find $y_p$.

We want to find $y_p$ so that $y_p'' - 3y_p' + 4y_p = 3e^{2t}$

So let's try $y_p = Ae^{2t}$. Then

$4Ae^{2t} - 6Ae^{2t} + 4Ae^{2t} = 3e^{2t}$

$2Ae^{2t} = 3e^{2t}$

$A = \frac{3}{2}$

$y_p = \frac{3}{2}e^{2t}$. 
Remark

1. The same idea holds when \( g(t) \) is of a different form.

   - **Exponential**: Guess \( y_p \) is proportional to some exponential

   - **Sine or Cosine**: Guess \( y_p \) is a linear combination of sine and cosine

   - **Degree \( n \) polynomial**: Guess \( y_p \) is an \( n \) degree polynomial

   - Same idea when you have a product/sum of these three types of functions

2. If \( g(t) = g_1(t) + g_2(t) \) and \( [\Sigma Y_1] = g_1(t) \) and \( [\Sigma Y_2] = g_2(t) \). Then \( Y_1 + Y_2 \) is
a solution to \( L[y] = g(t) \). That is, the determining of \( y_p \) can be broken up into several smaller problems when \( g(t) \) can be expressed as a sum.

Ex Find a particular solution of

\[ y'' - 3y' - 4y = 3e^{2t} + 2 \sin t - 8e^t \cos(2t). \]

We can split the problem up into

\[ y'' - 3y' - 4y = 3e^{2t} \quad (1) \]

\[ y'' - 3y' - 4y = 2 \sin t \quad (2) \]

\[ y'' - 3y' - 4y = -8e^t \cos(2t) \quad (3) \]

First, we guess that \( y_p = Ae^{2t} \). Now,
$4A e^{2t} - 6A e^{2t} - 4A e^{2t} = 3 e^{2t}$

$-6A e^{2t} = 3 e^{2t}$

$A = -\frac{1}{2}$.

So,

$y_{p_1} = -\frac{1}{2} e^{2t}.$

Consider (2). We guess

$y_{p_2} = A \cos t + B \sin t$.

So

$y''_{p_2} - 3y'_{p_2} - y_{p_2} = 2 \sin t$

$-A \cos t - B \sin t - 3(-A \sin t + B \cos t)$

$- 4(A \cos t + B \sin t) = 2 \sin t$

$(-A - 3B - 4A) \cos t + (-B + 3A - 4B) \sin t = 2 \sin t$

$(-5A - 3B) \cos t + (-5B + 3A) \sin t = 2 \sin t$. 


\[-5A - 3B = 0 \quad -5B + 3A = 2\]

\[A = \frac{3}{17} \quad \text{and} \quad B = -\frac{5}{17}\]

so

\[y_{p_2} = \frac{3}{17} \cos t - \frac{5}{17} \sin t\]

Finally, for (3), we guess

\[y_{p_3} = Ae^{t \cos 2t} + Be^{t \sin 2t}\]

Plug \(y_{p_3}\) into (3) and we find

\[-10A - 2B = -8 \quad \text{and} \quad -10B + 2A = 0\]

so

\[A = \frac{10}{13} \quad \text{and} \quad B = \frac{2}{13}\]

Therefore,

\[y_{p_3} = \frac{10}{13} e^{t \cos 2t} + \frac{2}{13} e^{t \sin 2t}\]
Now, putting everything together,

\[ y_p = -\frac{1}{2} e^{2t} + \frac{3}{17} \cos t - \frac{5}{17} \sin t + \frac{10}{13} e^{t} \cos (2t) \]
\[ + \frac{2}{13} e^{t} \sin (2t). \]

Remark: Sometimes the guess for the form of the particular solution may overlap with the homogeneous solution. In such cases, multiply the guess by \( t \).

Ex: \( y'' - 3y' - 4y = 2e^{t} \)

The characteristic equation is

\[ r^2 - 3r - 4 = 0 \]
\[ (r-4)(r+1) = 0 \]
\[ r = 4, \quad r = -1 \]
so \[ y_c = c_1 e^{4t} + c_2 e^{-t} \]

is the solution to the homogeneous problem.

For the nonhomogeneous equation, our initial guess would be \( Ae^{-t} \), but this overlaps with the homogeneous solution so we guess

\[ y_p = Ae^{-t}. \]

Now,

\[ y_p' = Ae^{-t} - Ae^{-t} \]

and

\[ y_p'' = -Ae^{-t} - Ae^{-t} + Ae^{-t} + Ae^{-t} = -2Ae^{-t} + Ae^{-t} \]

Now,
\[ y_p'' - 3y_p' - 4y_p = 2e^{-t} \]

\[-2Ae^{-t} + Abe^{-t} - 3Ae^{-t} + 3Ate^{-t} - 4Ate^{-t} = 2e^{-t} \]

\[-5Ae^{-t} = 2e^{-t} \]

\[ A = -\frac{2}{5} \]

So \[ y_p = -\frac{2}{5} t e^{-t} \].
§ 3.6 Variation of Parameters

Consider the second-order DE
\[ y'' + p(t) y' + q(t) y = g(t) \]
and assume we know the general solution of the homogeneous equation to be
\[ y_c = c_1 y_1 + c_2 y_2. \]

Now, we suppose
\[ y_p = u_1(t) y_1 + u_2(t) y_2 \]
where \( u_1 \) and \( u_2 \) are to-be-determined.
\[ y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2' \]

Note that we have 1 equation for two unknown functions, so there may be many choices of \( u_1 \) and \( u_2 \). We add
the condition that
\[ u_1' y_1 + u_2' y_2 = 0 \]
so
\[ y_p' = u_1 y_1' + u_2 y_2' \]
Next
\[ y_p'' = u_1 y_1'' + u_1 y_1' + u_2 y_2'' + u_2 y_2' \]
Plugging in to the DE,
\[ u_1 y_1'' + u_1 y_1' + u_2 y_2'' + p(u_1 y_1' + u_2 y_2') \]
\[ + q (u_1 y_1' + u_2 y_2) = g \]
\[ u_1 (y_1'' + py_1' + qy_1) + u_2 (y_2'' + py_2' + qy_2) \]
\[ + u_1 y_1' + u_2 y_2' = g \]
so
\[ u_1 y_1' + u_2 y_2' = g \]
Since \( y_1 \) and \( y_2 \) are solutions to the homogeneous equation, therefore, we have the system of equations

\[
\begin{align*}
  u_1'y_1 + u_2'y_2 &= 0 \\
  u_1'y_1 + u_2'y_2 &= g(t)
\end{align*}
\]

and in matrix form

\[
\begin{bmatrix}
y_1 & y_2 \\
y_1' & y_2'
\end{bmatrix}
\begin{bmatrix}
u_1' \\
u_2'
\end{bmatrix}
=
\begin{bmatrix}
0 \\
g(t)
\end{bmatrix}
\]

\[
\begin{bmatrix}
u_1' \\
u_2'
\end{bmatrix}
=
\begin{bmatrix}
y_1 & y_2 \\
y_1' & y_2'
\end{bmatrix}^{-1}
\begin{bmatrix}
0 \\
g(t)
\end{bmatrix}
\]

\[
=rac{1}{y_1y_2'-y_2y_1'}
\begin{bmatrix}
y_2' & -y_2 \\
-y_1' & y_1
\end{bmatrix}
\begin{bmatrix}
0 \\
g(t)
\end{bmatrix}
\]
\[
\begin{align*}
&= \frac{1}{W[y_1, y_2]} \begin{bmatrix}
-y_2 g(t) \\
y_1 g(t)
\end{bmatrix} \\
\text{and so} \\
&u_1 = \frac{1}{W[y_1, y_2]} (-y_2 g(t)) \\
&u_2 = \frac{1}{W[y_1, y_2]} (y_1 g(t))
\end{align*}
\]

Thus,

\[
\begin{align*}
&u_1 = \int \frac{-y_2 g(t)}{W[y_1, y_2]} \, dt + c_1 \\
&u_2 = \int \frac{y_1 g(t)}{W[y_1, y_2]} \, dt + c_2
\end{align*}
\]

and our general solution is
\[ y = c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2 \]

**Ex.** Find the general solution of

\[ y'' + 4y = 8 \tan t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2} \]

First, we find the solution to the homogeneous eqn.

\[ r^2 + 4 = 0 \]

\[ r = \pm 2i \]

\[ y_c = C_1 \cos(2t) + C_2 \sin(2t) \]

Now, we suppose

\[ y_p = u_1 \cos(2t) + u_2 \sin(2t) \]

and so
\[ y_p' = -2u_1 \sin(2t) + 2u_2 \cos(2t) \]
\[ + u_1' \cos(2t) + u_2' \sin(2t) \]

We assume
\[ u_1' \cos(2t) + u_2' \sin(2t) = 0 \]

So
\[ y_p' = -2u_1 \sin(2t) + 2u_2 \cos(2t) \]

Now
\[ y_p'' = -2u_1' \sin(2t) - 4u_1 \cos(2t) \]
\[ + 2u_2' \cos(2t) - 4u_2 \sin(2t) \]

Plugging \( y_p \) into the nonhomogeneous equation, we have
\[ y_p'' + 4y = 8 \tan t \]
\[ -2u_1' \sin(2t) - 4u_1 \cos(2t) + 2u_2' \cos(2t) \]
\[ - 4u_2 \sin(2t) + 4u_1 \cos(2t) + 4u_2 \sin(2t) = 8 \tan t \]
\[-2u_1 \sin(2t) + 2u_2 \cos(2t) = 8 \tan t\]

and so we have the system

\[
u_1' \cos (2t) + u_2' \sin (2t) = 0
\]

\[-2u_1' \sin(2t) + 2u_2' \cos(2t) = 8 \tan t\]

\[
\begin{bmatrix}
\cos(2t) & \sin(2t) \\
-2\sin(2t) & 2\cos(2t)
\end{bmatrix}
\begin{bmatrix}
u_1' \\ u_2'
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
8 \tan t
\end{bmatrix}
\]

\[
\begin{bmatrix}
u_1' \\ u_2'
\end{bmatrix}
= 
\frac{1}{2\cos^2(2t) + 2\sin^2(2t)}
\begin{bmatrix}
2\cos(2t) & -\sin(2t) \\
2\sin(2t) & \cos(2t)
\end{bmatrix}
\begin{bmatrix}
0 \\
8 \tan t
\end{bmatrix}
\]

\[
= \frac{1}{2}
\begin{bmatrix}
-8 \sin(2t) \tan t \\
8 \cos(2t) \tan t
\end{bmatrix}
\]

\[
u_1' = -4 \sin(2t) \tan t \\
u_2' = 4 \cos(2t) \tan t
\]