§ 3.7 Mechanical and Electrical Vibrations

Many physical problems can be described by the IVP

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y'_0.$$ 

How to interpret $a$, $b$, and $c$?

Mass on a Spring

- mass $m$ hanging from spring
- spring has unstretched length $l$

Gravitational force $F_g = mg$  
  $g$ is the gravitational acceleration
Spring restoring force \( F_s = -KL \) (Hooke's law)

- \( K \) is the spring constant
- \( L \) is the displacement from spring rest length \( x \)

The mass is in equilibrium, so

\[
F_g + F_s = 0
\]

\[
mg - KL = 0.
\]

For dynamics, we want to study the motion of the mass if it is initially displaced or is acted on by an external force.

Let \( u(t) \) be the displacement from equilibrium. Then Newton's law of motion says

\[
m u'' = f(t)
\]
Where $f(t)$ is the net force acting on the mass.

Forces to consider:

1. Gravity: $F_g = mg$

2. Spring force: $F_s = -k(L + u)$

3. Damping (air resistance, energy dissipation, friction, mechanical)
   - Acts in direction opposite motion
   - Assume viscous damping
     - Resistive force is proportional to speed of the mass

   $$F_d = -\gamma u'$$
   - $\gamma$ is the damping constant
4. Applied Force \( F(t) \)

Now, we have

\[
m u'' = mg - k (L + u) - 8 u'(t) + F(t)
\]

\[
m u'' + 8 u' + Ku = mg - k L + F(t)
\]

or

\[
m u'' + 8 u' + Ku = F(t)
\]

Since \( mg - k L = 0 \).

Finally, we need to specify an initial position and initial velocity of the mass

\[
u(0) = u_0, \quad u'(0) = v_0
\]

Ex: A mass weighing 4 lbs stretches a spring 2 inches. Suppose that the
mass is given an additional 6 inch positive displacement in positive direction and then released. The mass exerts a viscous resistance of 6 lbs when mass has a velocity of 3 ft/s.

Set up the initial value problem. Let $u$ be the displacement from equilibrium.

First, the mass is

$$m = \frac{4 \text{ lb}}{\frac{32 \text{ ft}}{\text{s}^2}} = \frac{1}{8} \text{ slug (}\frac{\text{lb} \cdot \text{s}^2}{\text{ft}}\text{)}$$

The damping constant $\gamma$ is

$$F_d = -\gamma u'$$

$$+6 = +\gamma \cdot 3$$
\[ \gamma = \frac{6 \text{ lb}}{3 \text{ ft/s}} = 2 \text{ lb/s} \text{ ft}^{-1} \]

The spring constant \( K \) is

\[ F_s = -KL \]

\[ -4 = F_s = -KL \]

\[ -4 = -K \cdot 2 \]

\[ K = \frac{4 \text{ lb}}{2 \text{ in}} \cdot \frac{12 \text{ in}}{1 \text{ ft}} = 24 \text{ lb/ft} \]

Thus, \( m = \frac{1}{8} \text{ slug} \), \( \gamma = 2 \text{ lb/s} \text{ ft}^{-1} \), and \( K = 24 \text{ lb/ft} \).

Therefore, our DE is

\[ \frac{1}{8} u'' + 2 u' + 24 u = 0 \]

and ICs are

\[ u(0) = \frac{1}{2} \text{ ft} ; \quad u'(0) = 0 \]
What are possible solution behaviors? (without external forcing)

\[ m u'' + \gamma u' + Ku = 0 \]

The characteristic equation is

\[ mr^2 + \gamma r + K = 0 \]

\[ \Gamma_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mK}}{2m} \]

Solution has 3 forms (real part of roots < 0)

\( \gamma^2 - 4mK > 0 \)

\[ u = c_1 e^{\Gamma_1 t} + c_2 e^{\Gamma_2 t} \quad (\text{over damped}) \]

\( \gamma^2 - 4mK = 0 \)

\[ u = (c_1 + c_2 t) e^{-\frac{\gamma t}{2m}} \quad (\text{critically damped}) \]
\[ y^2 - 4mk < 0 \]

\[ U = e^{-\frac{\lambda t}{2m}} (c_1 \cos (\mu t) + c_2 \sin (\mu t)) \]

where \[ \mu = \frac{\sqrt{4km - \lambda^2}}{2m} \]

**Note:** When \( y^2 - 4mk < 0 \) and if \( \lambda = 0 \), we get pure oscillations, simple harmonic motion.

- \( \mu \) is the natural frequency
- Period \( T = \frac{2\pi}{\mu} \)
- Maximum displacement is the amplitude of the motion.

**Note 2:** When \( \lambda \neq 0 \)

- \( \mu \) is called the quasi-frequency
$T$ is called the quasi-period

\[ \chi^2 - \Gamma \omega_n k < 0 \]

Additional application: Electrical circuits

§3.8 Forced Periodic Vibrations

We will now consider an external force $F(t)$, $m \ddot{u} + \gamma \dot{u} + ku = F(t)$

Suppose $F(t) = F_0 \cos(\omega t)$, i.e. a
periodic force of amplitude $F_0$ and frequency $\omega$. Then

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t)$$

and we expect the solution to be of the form

$$u = u_c + u_p$$

$$= c_1 u_1 + c_2 u_2 + A \cos(\omega t) + B \sin(\omega t)$$

We know $u_c \to 0$ as $t \to \infty$ since $w, \gamma, k > 0$.

- $u_c$ is called the transient solution
- $u_p$ persists indefinitely and is called the steady-state solution or the forced response of the system
The effect of the ICs is lost in the presence of damping.

Let's take a closer look at:

\[ U_p = A \cos(wt) + B \sin(wt) \]

After some manipulations, we may write:

\[ U_p = R \cos(wt - \theta) \]

where:

\[ R = \frac{E_0}{\Delta}, \quad \cos \theta = \frac{m(w_0^2 - \omega^2)}{\Delta} \]

and

\[ \sin \theta = \frac{\gamma \omega}{\Delta} \]

where:

\[ \Delta = \sqrt{m^2(w_0^2 - \omega^2)^2 + \gamma^2\omega^2} \]

and

\[ w_0^2 = \frac{K}{m} \]
Now, \[
\frac{R}{F_0/K} = \frac{K}{\Delta}
\]

\[
= \left( \left( 1 - \left( \frac{w}{w_0} \right)^2 \right)^2 + \frac{1}{\varpi} \left( \frac{w}{w_0} \right)^2 \right)^{-\frac{1}{2}}
\]

where \[\varpi = \frac{g^2}{mk}\]

\[\frac{R}{F_0/K}\] is the ratio of the force response amplitude to displacement caused by force \(F_0\).

\(w \to 0\) (low frequency limit)
\[
\frac{R}{F_0 K} \rightarrow 1 \quad \text{or} \quad R \rightarrow \frac{F_0}{K}
\]

\[w \rightarrow \infty \quad \text{(high frequency limit)}\]

\[R \rightarrow 0\]

There is an intermediate value of \(w\) where \(R\) has a maximum:

\[w_{\text{max}}^2 = w_0^2 \left(1 - \frac{x^2}{2w_0K}\right)\]

\[R_{\text{max}} = \frac{F_0}{8w_0 \sqrt{1 - \left(\frac{x^2}{2w_0K}\right)}}\]

For small \(x\):

\[R_{\text{max}} \approx \frac{F_0}{8w_0}\]

and as \(x\) gets smaller, \(R_{\text{max}}\) grows
This is called resonance.

Other areas to explore:

- Forced vibrations without damping
- Beats
- Amplitude modulation