

1. (10 pts.) Consider the following differential equation

$$3u'' + 4u' + 2u = 0.$$

What happens to solutions as $t \rightarrow \infty$?

2. (10 pts.) Compute e^{At} where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Express your answer as $\mathbf{u} = \mathbf{v} + i\mathbf{w}$ where \mathbf{v} and \mathbf{w} are real.

3. (15 pts.) Solve the following initial value problem

$$y'' - 3y' + 2y = (1 + e^{-t})^{-1}, \quad y(0) = 0, \quad y'(0) = 1.$$

Hint: $\int \frac{e^{-2t}}{1+e^{-t}} dt = \ln|1 + e^{-t}| - e^{-t}$.

4. (15 pts.) Use the Laplace transform to solve the following initial value problem

$$y'' + 2y' + 2y = 2, \quad y(0) = 0, \quad y'(0) = 1.$$

5. (10 pts.) Consider the following system of differential equations

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \mathbf{x}.$$

(a) (7 pts.) Write down the fundamental matrix, and show that the solutions you found in (a) form a fundamental set of solutions.

(b) (3 pts.) Suppose $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Determine the solution to the initial value problem.

6. (15 pts.) Find the general solution of the following system of differential equations

$$\mathbf{x}' = \begin{pmatrix} 3 & -4 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x}.$$

7. (15 pts.) Find the general solution of the following system of differential equations

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

8. (10 pts.) Consider the following differential equation

$$\frac{dy}{dt} = f(y) = (y - 1)(y^2 - 4).$$

- (a) (2 pts.) Find the equilibria and classify their stability
- (b) (2 pts.) Plot $f(y)$ vs. y and sketch the phase line.
- (c) (2 pts.) Plot solution trajectories in the y vs. t plane.
- (d) (4 pts.) Describe the behavior of solutions as $t \rightarrow \infty$.

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\frac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\frac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. t^n ; $n = \text{positive integer}$	$\frac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
4. t^p , $p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
5. $\sin at$	$\frac{a}{s^2 + a^2}, \quad s > 0$	Sec. 6.1; Ex. 6
6. $\cos at$	$\frac{s}{s^2 + a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\frac{a}{s^2 - a^2}, \quad s > a $	Sec. 6.1; Prob. 8
8. $\cosh at$	$\frac{s}{s^2 - a^2}, \quad s > a $	Sec. 6.1; Prob. 7
9. $e^{at} \sin bt$	$\frac{b}{(s-a)^2 + b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at} \cos bt$	$\frac{s-a}{(s-a)^2 + b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}$, $n = \text{positive integer}$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 19
16. $\int_0^t f(t-\tau)g(\tau) d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \dots - f^{(n-1)}(0)$	Sec. 6.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 28