1 Basic Models and Direction Fields

What is a differential equation? Simply put, it is an equation of derivatives. Why would one want to study such equations? As we will see, they are convenient ways of describing the physical processes that we observe. That is, we observe something happen, and we hope to model the process using mathematics.

Consider the following example. All across campus, we see Turkeys, squirrels, and ducks. During certain times of the year, we encounter them more often than other times of the year, so we may initially hypothesize that the populations we observe on campus are not constant throughout the year. Let $p(t)$ be the population of turkeys at time t. What are some ways that the population can change?

- Migration
- Breeding
- Predation
- Accidents

Those are a few ways that the population can change, and we can broadly categorize these as births and deaths. Here is a differential equation

$$\frac{dp}{dt} = 4p - 10.$$

1.1 Direction Fields

Consider the arbitrary differential equation (DE)

$$\frac{dy}{dt} = f(t, y).$$

When we evaluate f, this tells us the value of $\frac{dy}{dt}$ at a point (t, y). That is, $f(t, y)$ tells us the slope of the cuve $y(t)$ at the points (t, y) in the ty-plane.

What are some potential problems with this model?

- The model can produce negative populations
- The population goes to $\pm\infty$

Now, we go back to the drawing board and cook-up an improved model.

1.2 Building a Model

When building a model, the general steps to follow are

1. Identify independent and dependent variables
2. Choose convenient units
3. Describe what you are modeling and basic principles governing the problem
4. Translate what you are modeling and the basic principles to the independent and dependent variables you identified in step 1

2 Constructing and Solving Your First Differential Equation

Consider an object of mass m falling with air resistance whose velocity at time t is given by $v(t)$. Newton's second law states that $F = ma$, where F is the net force acting on the object, and $a = \frac{dv}{dt}$ is the object's acceleration. The forces acting on a falling object are the gravitational force and air resistance. The gravitational force is $F_g = mg$, where g is the gravitational acceleration. Since the object is falling, air resistance acts in the opposite direction of the gravitational force, and we will assume air resistance is proportional to velocity and acts in the opposite direction of velocity. Therefore, $f_{\text{air}} = -\gamma v$, where γ is some constant of proportionality. Thus, Newton's second law gives

$$m\frac{dv}{dt} = mg - \gamma v,$$

and we have arrived at a differential equation describing the velocity of a falling object subject to air resistance. Now, let us solve our first differential equation. We have

$$\begin{aligned} m\frac{dv}{dt} &= mg - \gamma v \\ \frac{dv}{dt} &= g - \frac{\gamma}{m}v \\ \frac{dv}{dt} &= -\frac{\gamma}{m}\left(v - \frac{gm}{\gamma}\right) \\ \frac{v'}{v - \frac{gm}{\gamma}} &= -\frac{\gamma}{m} \\ \frac{d}{dt}\left(\ln\left|v - \frac{gm}{\gamma}\right|\right) &= -\frac{\gamma}{m} \\ \int \frac{d}{dt}\left(\ln\left|v - \frac{gm}{\gamma}\right|\right)\, dt &= \int -\frac{\gamma}{m}\, dt \\ \ln\left|v - \frac{gm}{\gamma}\right| &= -\frac{\gamma}{m}t + C_1 \end{aligned}$$

where C_1 is some arbitrary constant resulting from the integration that we performed. Since we want to know the velocity of the falling object, we find, after some manipulation, that

$$v(t) = \frac{gm}{\gamma} - C_2 e^{-\frac{\gamma}{m}t}.$$

The above expression is called the general solution, and its geometric representation is a family of curves called integral curves.

We may be unsatisfied that we are left with the consant C_2 in our expression, but we may determine C_2 using an initial condition. Suppose the object is released at time $t = 0$. Then, our initial condition is $v(0) = 0$. A differential equation with an initial value is called an initial value problem. For our problem at hand, the initial value problem is

$$m\frac{dv}{dt} = mg - \gamma v, \quad v(0) = 0.$$

Using the initial condition, we find that

$$v(0) = 0 = \frac{gm}{\gamma} - C_2$$

so $C_2 = \frac{gm}{\gamma}$. Thus, the solution to the initial value problem is

$$v(t) = \frac{gm}{\gamma}\left(1 - e^{-\frac{\gamma}{m}t}\right).$$

Notice that as $t \to \infty$, $v \to \frac{gm}{\gamma}$. That is, the object approaches some terminal velocity.

3 Classification of Differential Equations

3.1 PDE or ODE?

The first classification of differential equations that we will consider is whether we are given an ordinary differential equation or a partial differential equation. An ordinary differential equation (ODE) is an equation involving derivatives with repect to only one variable. For example,

$$\frac{d^2Q}{dt^2} + \frac{dQ}{dt} + Q = 0$$

or

$$Q'' + Q' + Q = 0$$

is an ODE. When we have an ODE with higher order derivatives, we use the notation

$$Q^{(n)} = \frac{d^n Q}{dt^n},$$

so

$$Q^{(1)} = \frac{dQ}{dt} = Q'.$$

When our unknown function has more than one independent variable, we are dealing with a partial differential equation. That is, we have an equation which involves derivatives with respect to more than one variable. A classical example of a partial differential equation is the heat equation

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}.$$

3.2 Systems of Differential Equations

Next, instead of being given a single differential equation, we may be given a system of differential equations. These arise when we have multiple dependent variables that interact. For example, the Lotka-Volterra (predator-prey) system is,

$$\frac{dx}{dt} = ax - \alpha xy$$
$$\frac{dy}{dt} = -cy + \gamma xy,$$

where x is the prey population and y is the predator population. The terms αxy and γxy describe the interaction between the predators and prey.

3.3 Order of a Differential Equation

The order of a differential equation is the order of the highest derivative appearing in the equation. For example,

$$y''' + yy' = t^4$$

is a third order ODE, because the highest derivative appearing is a third order derivative. A general n^{th} order ODE is of the form

$$F\left(t, y, y', \ldots, y^{(n)}\right) = 0.$$

We will always assume that we may write the function F as

$$y^{(n)} = f\left(t, y, y', \ldots, y^{(n-1)}\right).$$

3.4 Linear and Nonlinear Equations

The ODE

$$F\left(t, y, y', \ldots, y^{(n)}\right) = 0$$

is called linear if F is linear in y and its derivatives. A general n^{th} order linear ODE is of the form

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t).$$

Notice that the differential equation does not need to be linear in the independent variable. If the differential equation is not linear, then we refer to the differential equation as nonlinear. Nonlinear equations are much harder to solve, but sometimes we can approximate them using linear equations by linearing the differential equation. For example, if our differential equation contained a term involving $\sin\theta$, and θ was our dependent variable, we could use the approximation $\sin\theta \approx \theta$ to linearize the differential equation.

3.5 Solutions of Differential Equations

Loosely speaking, a solution of a differential equation, is a function that satisfies the equation. More precisely, consider the interval $\alpha < t < \beta$ and the n^{th} order ODE

$$y^{(n)} = f\left(t, y, y', \ldots, y^{(n-1)}\right).$$

The function $\varphi(t)$ is called a solution of the ODE if

$$\varphi^{(n)} = f\left(t, \varphi, \varphi', \ldots, \varphi^{(n-1)}\right)$$

for every $t \in (\alpha, \beta)$.

Consider the following differential equation

$$ty' - y = t^2.$$

We claim that $\varphi(t) = 3t + t^2$ is a solution to the differential equation. Differentiating our candidate function, we see that $\varphi'(t) = 3 + 2t$. Now, we substitute our candidate function into the differential equation, and we have

$$\begin{aligned} t(3+2t) - (3t + t^2) &= t^2 \\ t^2 &= t^2. \end{aligned}$$

We see that the two sides agree, so the equation is satisfied. Thus, we conclude that our claim is valid, and $\varphi(t) = 3t + t^2$ is indeed a solution to the differential equation.

Finding solutions to a given differential equation is often a monumental task, but with enough experience, one can start using one of the most useful techniques for solving a differential equation: making an ansatz. An ansatz is an educated guess which you can make, and if your ansatz satisfies the differential equation, then you've found a perfectly valid and acceptable solution!

Consider the following differential equation

$$y'' + y = 0.$$

Since we may rewrite the differential equation as

$$y'' = -y,$$

we see that we are looking for a function that produces the negative of itself after two differentiations. Two familiar functions that satisfy this property are sine and cosine.

3.6 Some remaining questions

We have seen that we can pull solutions out of thin air, but we may not always be able to do this. Additionally, we may find that there is more than one function that satisfies the differential equation. Is this acceptable? Finally, when can we explicitly determine the solution of a differential equation. The questions that we will eventually answer are:

1. Given a differential equation, when does a solution exist?
2. Given a differential equation, how many solutions does it have? Is the solution unique?
3. Given a differential equation, can we actually determine a solution?