1 The Laplace Transform

1.1 Integral Preliminaries

Recall the improper integral

$$\int_a^\infty f(t)\,dt = \lim_{b\to\infty}\int_a^b f(t)\,dt.$$

If $\int_a^b f(t)\,dt$ exists for all $b > a$ and the limit as $b \to \infty$ exists, then the improper integral is said to converge. Otherwise, it diverges. Showing convergence directly may be difficult, so we use comparison tests.

Definition. *A function f is called piecewise continuous on an interval $\alpha \le t \le \beta$ if the interval can be partitioned by a finite number of points $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ so that*

1. *f is continuous on each subinterval $t_{i-1} < t < t_i$,*
2. *f approaches a finite limit at each endpoint of each subinterval as it is approached from within the subinterval.*

We may now write $\int_\alpha^\beta f(t)\,dt$ as

$$\int_\alpha^\beta f(t)\,dt = \int_\alpha^{t_1} f(t)\,dt + \int_{t_1}^{t_2} f(t)\,dt + \cdots + \int_{t_{n-1}}^{\beta} f(t)\,dt.$$

Theorem. *If f is piecewise continuous for $a \le t$, $|f(t)| \le g(t)$ for $t \ge M$ where M is some positive constant, and $\int_M^\infty g(t)\,dt$ converges, then $\int_a^\infty f(t)\,dt$ converges. On the other hand, if $f(t) \ge g(t) \ge 0$ for $t \ge M$ and $\int_M^\infty g(t)\,dt$ diverges, then $\int_a^\infty f(t)\,dt$ diverges.*

1.2 The Laplace Transform

An integral transform is an integral of the form

$$F(s) = \int_\alpha^\beta K(s,t) f(t)\,dt,$$

where $K(s,t)$ is a given function called the kernel of the transformation. Notice that after performing the integral, we are left with a function of s, and $F(s)$ is called the transform of $f(t)$.

Definition. *(Laplace Transform) The Laplace Transform is the integral transform with kernel $K(s,t) = e^{-st}$. That is,*

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt,$$

and we will denote the Laplace transform of $f(t)$ as $F(s)$.

Steps for using the Laplace transform to solve an initial value problem:

1. Transform the differential equation from t-domain to a simpler problem in the s-domain.
2. Solve the resulting algebraic problem to find $F(s)$.
3. Recover f from its transform F.

Theorem. *Suppose that f is piecewise continuous on $0 \le t \le b$ for any positive b, and there are constants k, a, and M so that*

$$|f(t)| \le k e^{at},$$

when $t \ge M$. Then, $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a$.

Proof. The Laplace Transform of f is

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_0^\infty f(t)e^{-st}\,dt \\ &= \int_0^M f(t)e^{-st}\,dt + \int_M^\infty f(t)e^{-st}\,dt.\end{aligned}$$

Now,

$$\int_0^M f(t)e^{-st}\,dt < \infty,$$

and

$$\begin{aligned}\left|\int_M^\infty f(t)e^{-st}\,dt\right| &\le \int_M^\infty |f(t)|\,e^{-st}\,dt \\ &\le \int_M^\infty ke^{at}e^{-st}\,dt \\ &= \int_M^\infty ke^{(a-s)t}\,dt \\ &= \left.\frac{ke^{(a-s)t}}{a-s}\right|_M^\infty \\ &= \lim_{t\to\infty}\frac{ke^{(a-s)t}}{a-s} - \frac{ke^{(a-s)M}}{a-s} \\ &= \frac{ke^{(a-s)M}}{s-a} + \lim_{t\to\infty}\frac{ke^{(a-s)t}}{a-s} \\ &< \infty\end{aligned}$$

when $a < s$. Thus, we have shown that $\mathcal{L}\{f(t)\}$ exists for $s > a$. □

1.3 Some Laplace Transforms

1.3.1 $\mathcal{L}\{c\}$

1. $f(t) = c$, c some constant

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_0^\infty f(t)e^{-st}\,dt \\ &= \int_0^\infty ce^{-st}\,dt \\ &= \left.\frac{ce^{-st}}{-s}\right|_0^\infty \\ &= 0 - \frac{c}{-s} \\ &= \frac{c}{s}.\end{aligned}$$

Thus,

$$\mathcal{L}\{f(t)\} = \frac{c}{s}.$$

2. $f(t) = e^{at}$

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_0^\infty e^{at}e^{-st}\,dt \\ &= \int_0^\infty e^{(a-s)t}\,dt \\ &= \left.\frac{e^{(a-s)t}}{a-s}\right|_0^\infty \\ &= 0 - \frac{1}{a-s} \\ &= \frac{1}{s-a}\end{aligned}$$

when $a < s$. Thus,

$$\mathcal{L}\{f(t)\} = \frac{1}{s-a}.$$

3. $f(t) = \sin(at)$

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_0^\infty \sin(at)e^{-st}\,dt \\ &= \left.\frac{e^{-st}\sin(at)}{-s}\right|_0^\infty - \int_0^\infty \frac{a}{-s}e^{-st}\cos(at)\,dt \\ &= 0 - 0 + \infty_0^\infty \frac{a}{s}e^{-st}\cos(at)\,dt \\ &= \frac{a}{s}\int_0^\infty e^{-st}\cos(at)\,dt \\ &= \frac{a}{s}\left(\left.\frac{e^{-st}\cos(at)}{-s}\right|_0^\infty - \int_0^\infty \frac{a}{s}e^{-st}\sin(at)\,dt\right) \\ &= \frac{a}{s}\left(0 - \frac{1}{-s} - \frac{a}{s}\int_0^\infty e^{-st}\sin(at)\,dt\right).\end{aligned}$$

Now, $F(s) = \mathcal{L}\{f(t)\}$, so

$$\begin{aligned}F(s) &= \frac{a}{s}\left(\frac{1}{s} - \frac{a}{s}F(s)\right) \\ F(s) + \frac{a^2}{s^2}F(s) &= \frac{a}{s^2} \\ F(s) &= \frac{a}{s^2}\frac{1}{1+\frac{a^2}{s^2}} \\ F(s) &= \frac{a}{s^2+a^2}.\end{aligned}$$

Thus,

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}.$$

4. $f(t) = \cos(at)$ (Similar process as 3.)

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}.$$

The Laplace Transform is a linear operator, so

$$\begin{aligned}\mathcal{L}\{c_1f_1(t)+c_2f_2(t)\} &= \int_0^\infty (c_1f_1(t)+c_2f_2(t))\,e^{-st}\,dt\\ &= \int_0^\infty c_1f_1(t)e^{-st}+c_2f_2(t)e^{-st}\,dt\\ &= c_1\int_0^\infty f_1(t)e^{-st}\,dt + c_2\int_0^\infty f_2(t)e^{-st}\,dt\\ &= c_1\mathcal{L}\{f_1(t)\}+c_2\mathcal{L}\{f_2(t)\}\\ &= c_1F_1(s)+c_2F_2(s).\end{aligned}$$

That is, if we want to compute the Laplace transform of some sum of functions, we can split the problem into finding the Laplace Transform of each individual function.

Example 1.1. *Determine the Laplace transform of*

$$f(t) = 5e^{-2t} - 3\sin(4t).$$

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\{5e^{-2t}-3\sin(4t)\}\\ &= 5\mathcal{L}\{e^{-2t}\} - 3\mathcal{L}\{\sin(4t)\}\\ &= \frac{5}{s+2} - \frac{12}{s^2+16}.\end{aligned}$$

2 Solution of Initial Value Problems

We will now see how the Laplace Transform may be used to solve initial value problems.

Theorem. *Suppose f is continuous and f' is piecewise continuous on $0 \le t \le b$. Suppose that there are constants k, a, and M such that $|f(t)| \le ke^{at}$ for $t \ge M$. Then $\mathcal{L}\{f'(t)\}$ exists and*

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

Proof. First, we write

$$\begin{aligned}\int_0^b e^{-st}f'(t)\,dt &= \int_{t_0}^{t_1} f'(t)e^{-st}\,dt + \int_{t_1}^{t_2} f'(t)e^{-st}\,dt + \cdots + \int_{t_{n-1}}^{t_n} f'(t)e^{-st}\,dt\\ &= \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} f'(t)e^{-st}\,dt\end{aligned}$$

since $f'(t)$ is piecewise continuous. Now,

$$\begin{aligned}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} f'(t)e^{-st}\,dt &= \sum_{k=0}^{n-1} f(t)e^{-st}\Big|_{t_k}^{t_{k+1}} + \int_{t_k}^{t_{k+1}} se^{-st}f(t)\,dt\\ &= f(t_1)e^{-st_1} - f(t_0)e^{-st_0} + f(t_2)e^{-st_2} - f(t_1)e^{-st_1} + \cdots + f(t_n)e^{-st_n} - f(t_{n-1})e^{-st_{n-1}}\\ &\quad + \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} se^{-st}f(t)\,dt\\ &= f(t_n)e^{-st_n} - f(t_0)e^{-st_0} + s\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} e^{-st}f(t)\,dt\\ &= f(b)e^{-sb} - f(0) + s\int_0^b e^{-st}f(t)\,dt.\end{aligned}$$

Now,

$$\lim_{b\to\infty}\left(f(b)e^{-sb}-f(0)+s\int_0^b e^{-st}f(t)\,dt\right)=s\int_0^\infty e^{-st}f(t)\,dt-f(0),$$

so we find that

$$\mathcal{L}\{f'(t)\}=s\mathcal{L}\{f(t)\}-f(0).$$

□

Now, let us use this result to compute $\mathcal{L}\{f''(t)\}$. We have

$$\begin{aligned}\mathcal{L}\{f''(t)\}&=s\mathcal{L}\{f'(t)\}-f'(0)\\&=s\left(s\mathcal{L}\{f(t)\}-f(0)\right)-f'(0)\\&=s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0).\end{aligned}$$

For $\mathcal{L}\{f^{(n)}(t)\}$, we have the following corollary.

Corollary. $\mathcal{L}\{f^{(n)}(t)\}=s^n\mathcal{L}\{f(t)\}-s^{(n-1)}f(0)-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$.

Example 2.1. *Use the Laplace transform to solve the initial value problem*

$$y''-y'-2y=0,\quad y(0)=1,\quad y'(0)=0.$$

First, we transform to the s-domain. Let $Y(s)=\mathcal{L}\{y(t)\}$. *Then*

$$\begin{aligned}\mathcal{L}\{y''-y'-2y\}&=\mathcal{L}\{0\}\\\mathcal{L}\{y''\}-\mathcal{L}\{y'\}-2\mathcal{L}\{y\}&=0.\end{aligned}$$

Now,

$$\begin{aligned}\mathcal{L}\{y''\}&=s^2\mathcal{L}\{y\}-sy(0)-y'(0)\\&=s^2Y(s)-s-0\\&=s^2Y(s)-s,\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}\{y'\}&=s\mathcal{L}\{y\}-y(0)\\&=sY(s)-1.\end{aligned}$$

Then, we have

$$s^2Y(s)-s-(sY(s)-1)-2Y(s)=0.$$

Now, we solve for $Y(s)$,

$$\begin{aligned}s^2Y(s)-s-sY(s)+1-2Y(s)&=0\\Y(s)\left(s^2-s-2\right)&=s-1\\Y(s)&=\frac{s-1}{s^2-s-2}.\end{aligned}$$

Thus, we have

$$Y(s)=\frac{s-1}{s^2-s-2}.$$

We have found the solution in the s-domain. The final step is to go back to the t-domain.

Notice that our differential equation became an algebraic equation. This vastly simplified the problem, but we are now left with returning to the t-domain.