1 Inverting the Laplace Transform

Rather than explicitly using the inverse Laplace Transform to return to the t-domain, we will compare the solution $Y(s)$ to known transformed functions to infer the inversion. Again, the key idea is linearity.

Suppose

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s),$$

and we know that $f_1(t) = \mathcal{L}^{-1}\{F_1(s)\}, \ldots, f_n(t) = \mathcal{L}^{-1}\{F_n(s)\}$. Then,

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F_1(s)\} + \cdots + \mathcal{L}^{-1}\{F_n(s)\}.$$

Example 1.1. *Previously, we found the solution to an initial value problem to be*

$$Y(s) = \frac{s-1}{s^2 - s - 2},$$

so let us now go back to the t-domain. Some Laplace transforms that we are familiar with are

- $\mathcal{L}\{c\} = \frac{c}{s}$
- $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$
- $\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$
- $\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$

Using partial fractions to rewrite $Y(s)$, we have

$$\begin{aligned} Y(s) &= \frac{s-1}{(s-2)(s+1)} \\ &= \frac{A}{s-2} + \frac{B}{s+1} \\ &= \frac{A(s+1) + B(s-2)}{(s-2)(s+1)} \\ &= \frac{s(A+B) + A - 2B}{(s-2)(s+1)}, \end{aligned}$$

so we require

$$\begin{aligned} A + B &= 1 \\ A - 2B &= -1. \end{aligned}$$

Therefore, $A = \frac{1}{3}$ and $B = \frac{2}{3}$. Thus,

$$Y(s) = \frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1}.$$

Now, comparing $Y(s)$ to our list of known Laplace transforms, we see that

$$\begin{aligned} Y(s) &= \frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1} \\ \mathcal{L}^{-1}\{Y(s)\} &= \mathcal{L}^{-1}\left\{\frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1}\right\} \\ y(t) &= \mathcal{L}^{-1}\left\{\frac{\frac{1}{3}}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{\frac{2}{3}}{s+1}\right\} \\ y(t) &= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \\ y(t) &= \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}. \end{aligned}$$

We have found that

$$y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}.$$

2 Systems of First-Order Linear Equations: Introduction

We will consider systems of ODEs and denote the independent variable by t, the dependent variables by x_1, x_2, $\ldots$, x_n, and derivatives by $\frac{dx_1}{dt}$, x_1', or $\dot{x}_1$. Why study systems of ODEs? Your model may involve involve interacting quantities, you may arrive at a system of ODEs upond discretizing a PDE, or you may have an equation involving higher order derivatives. We will see that higher order differential equations can be reformulated as a system of first-order ODEs.

Example 2.1. *Recall the general equation for a spring-mass system*

$$mu'' + \gamma u' + ku = F(t).$$

Let $x_1 = u$ and $x_2 = u'$. Then,

$$x_1' = u'$$

and

$$x_2' = u'',$$

so

$$\begin{aligned} x_2' &= u'' \\ &= \frac{1}{m}\left(F(t) - \gamma u' - ku\right) \\ &= \frac{1}{m}\left(F(t) - \gamma x_2 - kx_1\right). \end{aligned}$$

In matrix form, we have the system

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{F(t)}{m} \end{bmatrix}.$$

For an arbitrary n^{th} order equation,

$$y^{(n)} = F(t, y, y', \ldots, y^{(n-1)}),$$

we let

$$\begin{aligned} x_1 &= y \\ x_2 &= y' \\ &\vdots \\ x_n &= y^{(n-1)}, \end{aligned}$$

so that

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = x_3 \\ &\vdots \\ x_n' &= y^{(n)} = F(t, x_1, x_2, \ldots, x_n). \end{aligned}$$

More generally, the system of n first-order ODEs is

$$\begin{aligned} x_1' &= F_1(t, x_1, x_2, \ldots, x_n) \\ x_2' &= F_2(t, x_1, x_2, \ldots, x_n) \\ &\vdots \\ x_n' &= F_n(t, x_1, x_2, \ldots, x_n), \end{aligned}$$

and the solution of such a system consists of n functions

$$\begin{aligned} x_1 &= \varphi_1(t) \\ x_2 &= \varphi_2(t) \\ &\vdots \\ x_n &= \varphi_n(t) \end{aligned}$$

to which we may also prescribe initial conditions

$$\begin{aligned} x_1(t_0) &= x_1^0 \\ x_2(t_0) &= x_2^0 \\ &\vdots \\ x_n(t_0) &= x_n^0. \end{aligned}$$

We can think of the solutions as a set of parametric equations in n-dimensional space.

Theorem. *(Existence and Uniqueness) Let each of the n functions F_1, F_2, ..., F_n and the n^2 partial derivatives $\frac{\partial F_1}{\partial x_1}$, $\frac{\partial F_1}{\partial x_2}$, ..., $\frac{\partial F_1}{\partial x_n}$, ..., $\frac{\partial F_n}{\partial x_1}$, $\frac{\partial F_n}{\partial x_2}$, ..., $\frac{\partial F_n}{\partial x_n}$ be continuous in a region R of $tx_1x_2\cdots x_n$-space defined by $\alpha < t < \beta$, $\alpha_1 < x_1 < \beta_1$, ..., $\alpha_n < x_n < \beta_n$, and let $(t_0, x_1^0, x_2^0, \ldots, x_n^0) \in R$. Then there is an interval $|t_0 - t| < h$ in which there exists a unique solution $\vec{x} = \vec{\varphi}$ to the system of ODEs that satisfies the initial condition.*

2.1 Linear and Nonlinear System

If F_1, ..., F_n are linear functions of x_1, ..., x_n, then the system is called linear. Otherwise, it is nonlinear. A general system of n first-order linear ODEs is of the form

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t) \\ x_2' &= p_{21}(t)x_1 + \cdots + p_{2n}(t)x_n + g_2(t) \\ &\vdots \\ x_1' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t) \end{aligned}$$

If $g_i(t) = 0$, $i = 1, \ldots, n$, for $t \in I$, the system is called homogeneous. Otherwise, it is called nonhomogeneous.

Theorem. *(Existence and Uniqueness for Linear Systems) If $p_{ij}(t)$ and $g_i(t)$, $i, j = 1, \ldots, n$, are continuous on $I : \alpha < t < \beta$, then there exists a unique solution $\vec{x} = \vec{\varphi}$ that satisfies the initial conditions. Moreover, the solution exists throughout the interval I.*

We will spend the remainder of our time studying linear systems of first-order ODEs, but first we review some of the tools we require from linear algebra.

3 Linear Algebra Review

3.1 Matrices

The $m \times n$ matrix with entires a_{ij} is

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

The transpose of A is denoted A^T, and if $A = (a_{ij})$, then $A^T = (a_{ji})$. The conjugate of A is denoted $\overline{A}$ and $\overline{A} = (\overline{a}_{ij})$. The adjoint of A is $\overline{A}^T$ or $A^* = (\overline{a}_{ji})$.

3.2 Properties of Matrices

1. Equality: A and B are equal if $a_{ij} = b_{ij}$ for each i and j.

2. Zero: $\vec{0}$ will denote the vector whose entries are all zero.

3. Addition:
$$A + B = (a_{ij}) + (b_{ij}).$$

4. Scalar Multiplication:
$$\alpha A = \alpha(a_{ij}) = (\alpha a_{ij})$$

5. Matrix Multiplication: $C = AB$,
$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$
Note that matrix multiplication is not commutative $AB \neq BA$

6. Multiplication of Vectors (dot product):
$$\vec{x}^T \vec{y} = \sum_{i=1}^{n} x_i y_i.$$
Note that this produces a scalar.
   - We may also consider the notion of an inner product,
   $$(\vec{x}, \vec{y}) = \sum_{i=1}^{n} x_i \overline{y}_i,$$
   - The length/magnitude of $\vec{x}$ is defined to be
   $$\|x\| = \sqrt{(\vec{x}, \vec{x})}.$$
   - If $(\vec{x}, \vec{x}) = 0$, then we must have that $\vec{x} = \vec{0}$.
   - If $(\vec{x}, \vec{y}) = 0$, then $\vec{x}$ and $\vec{y}$ are said to be orthogonal.

7. Identity matrix:
$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{pmatrix}$$
so that
$$AI = IA = A.$$

8. Inverse and Determinant:
   - A is called nonsingular/invertible if there exists a matrix B so that $AB = I$ and $BA = I$. We write $B = A^{-1}$.
   - If A has no inverse, we call A singular or noninvertible.
   - A is nonsingular if and only if $\det A \neq 0$.

3.3 Matrix Functions

We will consider vectors and matrices whose elements are functions.

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

and

$$A(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{m1}(t) & \cdots & a_{mn}(t) \end{pmatrix}.$$

$A(t)$ is called continuous if each element of A is continuous. Similarly,

$$\frac{dA}{dt} = \left(\frac{da_{ij}}{dt}\right),$$

and

$$\int_a^b A(t)\, dt = \left(\int_a^b a_{ij}\, dt\right).$$

Example 3.1. *Consider the matrix function*

$$A(t) = \begin{pmatrix} \sin t & t \\ 1 & \cos t \end{pmatrix}.$$

We see that

$$A'(t) = \begin{pmatrix} \cos t & 1 \\ 0 & -\sin t \end{pmatrix},$$

and

$$\begin{aligned} \int_0^\pi A(t)\, dt &= \begin{pmatrix} -\cos t & \frac{t^2}{2} \\ t & \sin t \end{pmatrix}\Bigg|_0^\pi \\ &= \begin{pmatrix} 2 & \frac{\pi^2}{2} \\ \pi & 0 \end{pmatrix}. \end{aligned}$$

Many of our familiar rules from calculus hold,

- $\frac{d}{dt}(CA) = C\frac{dA}{dt}$, where C is a constant matrix
- $\frac{d}{dt}(A+B) = \frac{dA}{dt} + \frac{dB}{dt}$
- $\frac{d}{dt}(AB) = A\frac{dB}{dt} + \frac{dA}{dt}B$.

3.4 Systems of Linear Algebraic Equations

A system of linear algebraic equations is of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{n1}x_n + a_{n2}x_n + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

or

$$A\vec{x} = \vec{b}.$$

If $\vec{b} = \vec{0}$, the system is called homogeneous. Otherwise, it is nonhomogeneous. If A is nonsingular, then $\vec{x} = A^{-1}\vec{b}$, and the homogeneous problem only has the solution $\vec{x} = \vec{0}$. If A is singular, then $A\vec{x} = \vec{b}$ has no solution or a nonunique solution. The homogeneous problem $A\vec{x} = \vec{0}$ has infinitely many solution.

If A is singular, when does $A\vec{x} = \vec{b}$ have a solution? We must have that $(\vec{b}, \vec{y}) = 0$ for all $\vec{y}$ satsifying $A^*\vec{y} = \vec{0}$, where A^* is the adjoint of A. If this condition is satsifed, then there are infinitely many solutions of the form

$$\vec{x} = \vec{x}^{(0)} + \vec{\xi},$$

where $A\vec{x}^{(0)} = \vec{b}$ and $A\vec{\xi} = \vec{0}$.

3.5 Linear Dependence and Independence

We call $\vec{x}^{(1)}$,...,$\vec{x}^{(k)}$ linearly dependent if there are scalars $c_1, \ldots, c_k$, with at least one nonzero, such that

$$c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} + \cdots + c_k\vec{x}^{(k)} = \vec{0}.$$

If the only solution is $c_1 = c_2 = \cdots = c_k = 0$, then $\vec{x}^{(1)}$, ..., $\vec{x}^{(k)}$ are said to belinearly independent.

3.6 Eigenvalues and Eigenvectors

We may consider $A\vec{x} = \vec{y}$ as a linear transformation that maps $\vec{x}$ to $\vec{y}$. We are interested in finding the vectors which are mapped to a scalar multiple of itself, $\vec{y} = \lambda\vec{x}$. Then,

$$\begin{aligned} A\vec{x} &= \lambda\vec{x} \\ A\vec{x} - \lambda\vec{x} &= 0 \\ (A - \lambda I)\,\vec{x} &= 0. \end{aligned}$$

This equation has nonzero solutions if and only if $\det(A - \lambda I) = 0$, which we call the characteristic equation. We call λ an eigenvalue and $\vec{x}$ its corresponding eigenvector.