1 Basic Theory of Systems of First-Order Linear Equations

Consider the general linear system

$$\vec{x}'(t) = P(t)\vec{x} + \vec{g}(t),$$

where P is an $n \times n$ matrix. Continuity of P and $\vec{g}$ guarantees existence of solutions, and we will first consider the homogeneous equation

$$\vec{x}' = P(t)\vec{x}.$$

Later, we will consider the nonhomogeneous equation.

Theorem. *(Principle of Superposition) If $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ are solutions of $\vec{x}' = P(t)\vec{x}$, then $c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)}$ is a solution for any constants c_1 and c_2.*

If $\vec{x}^{(1)}, \ldots, \vec{x}^{(k)}$ are solutions to a system of k linear equations, then

$$\vec{x} = c_1\vec{x}^{(1)} + \cdots + c_k\vec{k}^{(k)}$$

is a solution. That is to say, every finite linear combination of solutions is a solution.

Consider the matrix

$$X(t) = \begin{pmatrix} | & | & & | \\ \vec{x}^{(1)} & \vec{x}^{(2)} & \cdots & \vec{x}^{(n)} \\ | & | & & | \end{pmatrix},$$

where $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ are solutions to the homogeneous system. The columns of $X(t)$ are linearly independent if and only if $\det(X(t)) \neq 0$. The determinant of X is called the Wronskian of $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ and denoted $W[\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(n)}](t)$.

Theorem. *If $\vec{x}^{(1)},\ldots,\vec{x}^{(n)}$ are linearly independent solutions, then the linear combination*

$$\vec{x} = c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t) + \cdots + c_n\vec{x}_n(t),$$

expresses each solution of the system $\vec{x}' = P(t)\vec{x}$ in exactly one way. If $c_1, \ldots, c_n$ are thought of as arbitrary, then $\vec{x}$ is called the general solution. A set of linearly independent solutions is said to form a fundamental set of solutions.

Theorem. *(Abel's Theorem) If $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ are solutions on $\alpha < t < \beta$, then $W[\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}](t)$ is either identically zero or never vanishes.*

Theorem. *If $\vec{x} = \vec{u}(t) + i\vec{v}(t)$ is a complex-valued solution, then its real part $\vec{u}(t)$ and its imaginary part $\vec{v}(t)$ are also solutions.*

2 Homogeneous Linear Systems with Constant Coefficients

Consider the system

$$\vec{x}' = A\vec{x},$$

where A is a constant $n \times n$ matrix. The equilibria of the system are where $\vec{x}' = 0$, so solutions to the homogeneous problem, $A\vec{x} = 0$, yield the equilibria. If $\det(A) \neq 0$, then $\vec{x} = 0$ is the only equilibria, so if A is nonsingular, the origin is our only equilibria. What about the stability of $\vec{x} = 0$? When $n = 2$, we can visualize qualitative features in the x_1x_2-plane (phase plane). We will use $A\vec{x}$ to plot the direction field and a number of solution curves (phase portrait).

2.1 Solvng Systems of ODEs

Consider the system

$$\vec{x}' = A\vec{x}.$$

Once again, we look for solutions of the form

$$\vec{x} = \vec{\xi}e^{rt}.$$

Then,

$$\begin{aligned} \vec{\xi} e^{rt} r &= A\vec{\xi} e^{rt} \\ A\vec{\xi} e^{rt} - \vec{\xi} e^{rt} r &= 0 \\ (A - rI)\,\vec{\xi} e^{rt} &= 0. \end{aligned}$$

Since $e^{rt} \neq 0$, we must have that

$$(A - rI)\vec{\xi} = 0.$$

Thus, we may determine the solutions of the system by finding the eigenpairs of A.

Example 2.1. *Find the general solution to*

$$\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x}.$$

First, we find the eigenvalues of the coefficient matrix. We have

$$\begin{aligned} \det\left(\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - \lambda I\right) &= 0 \\ \det\left(\begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix}\right) &= 0 \\ (1-\lambda)(1-\lambda) - 4 &= 0 \\ \lambda^2 - 2\lambda - 3 &= 0 \\ (\lambda - 3)(\lambda + 1) &= 0, \end{aligned}$$

so the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -1$. Now, we find the corresponding eigenvectors. Let $\vec{\xi_1} = \begin{pmatrix} \xi_{11} \\ \xi_{12} \end{pmatrix}$, so

$$\begin{aligned} (A - \lambda_1 I)\vec{\xi_1} &= \vec{0} \\ \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} \xi_{11} \\ \xi 12 \end{pmatrix} &= \vec{0}. \end{aligned}$$

We see that $\xi_{12} = 2\xi_{11}$. Now,

$$\vec{\xi} = \begin{pmatrix} \xi_{11} \\ \xi_{12} \end{pmatrix} = \begin{pmatrix} \xi_{11} \\ 2\xi_{11} \end{pmatrix} = \xi_{11} \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

so our eigenvector is $\vec{\xi_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Next, let $\vec{\xi_2} = \begin{pmatrix} \xi_{21} \\ \xi_{22} \end{pmatrix}$, so

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} \xi_{21} \\ \xi_{22} \end{pmatrix} = \vec{0}.$$

We see that $\xi_{22} = -2\xi_{21}$, so

$$\vec{\xi_2} = \begin{pmatrix} \xi_{21} \\ \xi_{22} \end{pmatrix} = \begin{pmatrix} \xi_{21} \\ -2\xi_{21} \end{pmatrix} = \xi_{21} \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

so our eigenvector is $\vec{\xi_2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Thus our solutions are

$$\vec{x}^{(1)} = e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{x}^{(2)} = e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

and the general solution is

$$\vec{x} = c_1 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

2.2 Drawing the Phase Plane

Often, solving a system of (nonlinear) differential equations my be infeasible, so plotting the phase plane will give us invaluable qualitative information which allows us to analyze the behavior of solutions. The information we will gather to paint our portrait are

1. Equilibria
2. Eigenvalues and eigenvectors
3. Nullclines
4. The sign of x_i' in regions defined by the nullclines

Example 2.2. *Sketch the phase plane for the system*

$$\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x}.$$

2.3 General $n \times n$ Systems

When solving a system of $n \times n$ system of linear first-order differential equations, we follow the same procedure:

1. Find eigenvectors and eigenvalues
2. Write down the general Solution
3. Use initial conditions to determine $c_1, \ldots, c_n$

The eigenvalues are determined by the n^{th}-degree polynomial equation

$$\det(A_\lambda I) = 0.$$

The eigenvalues that we may enounter come in three flavors:

1. real and distinct
2. complex conjugate pairs
3. repeated roots

Example 2.3. *Find the general solution to*

$$\vec{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \vec{x}.$$

First, we find the eigenvalues of the coefficient matrix. We have

$$\begin{aligned}
0 &= \det\left(\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} - \lambda I\right) \\
&= \det\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} \\
&= -\lambda \det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} - 1\begin{pmatrix} 1 & 1 \\ 1 & -\lambda \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ -\lambda & 1 \end{pmatrix} \\
&= -\lambda(\lambda^2 - 1) - (-\lambda - 1) + (1 + \lambda) \\
&= -\lambda(\lambda - 1)(\lambda + 1) + (\lambda + 1) + (1 + \lambda) \\
&= (\lambda + 1)\left(-\lambda(\lambda - 1) + 2\right) \\
&= (\lambda + 1)(-\lambda^2 + \lambda + 2) \\
&= (\lambda + 1)(-\lambda - 1)(\lambda - 2),
\end{aligned}$$

so our eigenvalues are $\lambda_{1,2} = -1$ *and* $\lambda_3 = 2$*. Now, we find the corresponding eigenvectors. Let* $\vec{\xi_1} = \begin{pmatrix} \xi_{11} \\ \xi_{12} \\ \xi_{13} \end{pmatrix}$*, so*

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_{11} \\ \xi_{12} \\ \xi_{13} \end{pmatrix} = \vec{0}.$$

We see that $\xi_{11} = -\xi_{12} - \xi_{13}$*, so*

$$\vec{\xi_1} = \begin{pmatrix} \xi_{11} \\ \xi_{12} \\ \xi_{13} \end{pmatrix} = \begin{pmatrix} -\xi_{12} - \xi_{13} \\ \xi_{12} \\ \xi_{13} \end{pmatrix} = \begin{pmatrix} -\xi_{12} \\ \xi_{12} \\ 0 \end{pmatrix} + \begin{pmatrix} -\xi_{13} \\ 0 \\ \xi_{13} \end{pmatrix} = \xi_{12} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \xi_{13} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore our two eigenvectors are

$$\vec{\xi_1} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{\xi_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Now, let $\vec{\xi_3} = \begin{pmatrix} \xi_{31} \\ \xi_{32} \\ \xi_{33} \end{pmatrix}$*, so*

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} \xi_{31} \\ \xi_{32} \\ \xi_{33} \end{pmatrix} = \vec{0}$$
$$\begin{pmatrix} 1 & -0.5 & -0.5 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} \xi_{31} \\ \xi_{32} \\ \xi_{33} \end{pmatrix} = \vec{0}$$
$$\begin{pmatrix} 1 & -0.5 & -0.5 \\ 0 & -1.5 & 1.5 \\ 0 & 1.5 & -1.5 \end{pmatrix} \begin{pmatrix} \xi_{31} \\ \xi_{32} \\ \xi_{33} \end{pmatrix} = \vec{0},$$

and $\xi_{31} = 0.5\xi_{32} + 0.5\xi_{33}$ *and* $\xi_{32} = \xi_{33}$*. Therefore, we see that*

$$\vec{\xi_3} = \begin{pmatrix} \xi_{31} \\ \xi_{32} \\ \xi_{33} \end{pmatrix} = \begin{pmatrix} 0.5\xi_{32} + 0.5\xi_{33} \\ \xi_{32} \\ \xi_{32} \end{pmatrix} = \begin{pmatrix} 0.5\xi_{32} + 0.5\xi_{32} \\ \xi_{32} \\ \xi_{32} \end{pmatrix} = \xi_{32} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

so $\vec{\xi_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$*. Thus, our solutions are*

$$\vec{x}^{(1)} = e^{-t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{x}^{(2)} = e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{x}^{(3)} = e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

and the general solution is

$$\vec{x} = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$