1 First-Order Differential Equations

We will now focus our attention on differential equations of the form

$$\frac{dy}{dt} = f(t, y).$$

1.1 Linear Differential Equations and the Method of Integrating Factors

Our most general first-order linear ODE is

$$\frac{dy}{dt} + p(t)y = g(t).$$

Notice that we have written the differential equation so that it is in standard form. Here, p and g are given functions. Sometimes we write

$$P(t)\frac{dy}{dt} + Q(t)y = G(t).$$

By using the product rule, we can rewrite the differential equation in a form that will allow us to arrive at the solution upon direct integration.

Consider the following differential equation

$$\sin(t)\frac{dy}{dt} + \cos(t)\,y = t.$$

Upon recognizing the product rule on the left-hand side, we have

$$\begin{aligned}
\sin(t)\frac{dy}{dt} + \cos(t)\,y &= t \\
\frac{d}{dt}\left(\sin(t)\,y\right) &= t \\
\int \frac{d}{dt}\left(\sin(t)\,y\right)\, dt &= \int t\, dt \\
\sin(t)\,y &= \frac{t^2}{2} + C \\
y &= \frac{\frac{t^2}{2} + C}{\sin(t)}
\end{aligned}$$

We are not always so lucky and able to directly factor using the product rule. Multiplying the differential equation by an integrating factor $\mu(t)$ will allow us to rewrite the differential equation and integrate directly.

1.2 Derivation of the integrating factor

We have the first order linear ODE

$$\frac{dy}{dt} + p(t)y = g(t)$$

to which we will multiply our to be determined integrating factor $\mu(t)$.

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t).$$

We hope that the left-hand side may be written as $\frac{d}{dt}\left(r(t)y\right)$, and computing this derivative, we see that

$$\frac{d}{dt}\left(r(t)y\right) = r(t)\frac{dy}{dt} + r'(t)y.$$

Therefore, we want

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = r(t)\frac{dy}{dt} + r'(t)y,$$

and comparing the two sides, we see that requiring $r(t) = \mu(t)$ and $r'(t) = \mu(t)p(t)$ will give us the desired result. Now, we may determine $\mu(t)$ by solving the differential equation

$$\mu'(t) = \mu(t)p(t).$$

After staring at the above equation, we see it is of the same form as the very first differential equation we solved. Following the same steps, we have

$$\begin{aligned} \mu'(t) &= \mu(t)p(t) \\ \frac{\mu'(t)}{\mu(t)} &= p(t) \\ \int \frac{d}{dt} \ln \mu(t)\, dt &= \int p(t)\, dt \\ \ln \mu(t) &= \int p(t)\, dt \\ \mu(t) &= e^{\int p(t)\, dt} \end{aligned}$$

Now, we turn our attention back to the differential equation. After multiplying the differential equation by our integrating factor, we have

$$\begin{aligned} e^{\int p(t)\, dt}\frac{dy}{dt} + e^{\int p(t)\, dt} p(t) y &= e^{\int p(t)\, dt} g(t) \\ \int \frac{d}{dt}\left(e^{\int p(t)\, dt} y\right)\, dt &= \int e^{\int p(t)\, dt} g(t)\, dt \\ e^{\int p(t)\, dt} y &= \int e^{\int p(t)\, dt} g(t)\, dt \\ y &= e^{-\int p(t)\, dt} \int e^{\int p(t)\, dt} g(t)\, dt, \end{aligned}$$

and so our solution is

$$y = e^{-\int p(t)\, dt} \int e^{\int p(t)\, dt} g(t)\, dt.$$

Some remarks are in order:

1. There will be a constant of integration resulting from computing $\int e^{\int p(t)\, dt} g(t)\, dt$, so when we solve a specific differential equation, our solution will look like
$$y = \psi(t)e^{-\int p(t)\, dt} + Ce^{-\int p(t)\, dt},$$
where $\psi(t) = \int e^{\int p(t)\, dt} g(t)\, dt$.
2. Notice that we did not concern ourselves with the constant of integration that should have appeared when we solved the differential equation for $\mu'(t)$. Do you see why we were allowed to ignore it?
3. The solution we arrived at may simply be memorized and applied, but I hope that the story told by our derivation of the integrating factor illustrates the motivation behind the technique.

Now, let us apply the method of integrating factors to the following initial value problem

$$(t^2 + 1)y' + y = t, \quad y(0) = 1.$$

Rewriting the differential equation in standard form, we have

$$y' + \frac{y}{t^2 + 1} = \frac{t}{t^2 + 1}.$$

Now, our integrating factor is

$$\begin{aligned} \mu(t) &= e^{\int \frac{1}{t^2+1}\, dt} \\ &= e^{\tan^{-1}(t)}. \end{aligned}$$

Multiplying the differential equation by $\mu(t)$, we have

$$\begin{aligned} e^{\tan^{-1}(t)}y' + \frac{e^{\tan^{-1}(t)}}{t^2+1}y &= \frac{t}{t^2+1}e^{\tan^{-1}(t)} \\ \frac{d}{dt}\left(e^{\tan^{-1}(t)}y\right) &= \frac{t}{t^2+1}e^{\tan^{-1}(t)} \\ \int \frac{d}{dt}\left(e^{\tan^{-1}(t)}y\right)\,dt &= \int \frac{t}{t^2+1}e^{\tan^{-1}(t)}\,dt \\ e^{\tan^{-1}(t)}y &= \int_0^t \frac{s}{s^2+1}e^{\tan^{-1}(s)}\,ds + C \\ y &= e^{-\tan^{-1}(t)}\int_0^t \frac{s}{s^2+1}e^{\tan^{-1}(s)}\,ds + Ce^{-\tan^{-1}(t)}. \end{aligned}$$

Now, to determine C, we use the initial condition which says that $y(0) = 1$, so

$$1 = y(0) = 0 + C.$$

Thus, the solution to the initial value problem is

$$y = e^{-\tan^{-1}(t)}\int_0^t \frac{s}{s^2+1}e^{\tan^{-1}(s)}\,ds + e^{-\tan^{-1}(t)}.$$

Remark: We see that we are left with an integral in our solution. This may seem unsatisfactory, but we may employ the use of a computer to approximate the definite integral which yields an approximation to the function y.

2 Separable Differential Equations

Consider the general first order ODE

$$\frac{dy}{dx} = f(x,y)$$

with initial condition $y(x_0) = y_0$. Suppose that we may write f as the product of two functions $g(x)$ and $h(y)$. That is, suppose we can separate f into two functions such that one of the functions only depends on x, and the other function only depends on y. Then, we have that

$$\begin{aligned} \frac{dy}{dx} &= f(x,y) \\ \frac{dy}{dx} &= g(x)h(y) \\ \frac{1}{h(y)}\frac{dy}{dx} &= g(x) \\ -g(x) + \frac{1}{h(y)}\frac{dy}{dx} &= 0. \end{aligned}$$

Now, let $G'(x) = -g(x)$ and $H'(y) = \frac{1}{h(y)}$. Then, we have

$$G'(x) + H'(y)\frac{dy}{dx} = 0,$$

and from the chain rule, we see that

$$\frac{d}{dx}\left(G(x) + H(y)\right) = G'(x) + H'(y)\frac{dy}{dx}.$$

Therefore,

$$
\begin{aligned}
G'(x) + H'(y)\frac{dy}{dx} &= 0 \\
\frac{d}{dx}\left(G(x) + H(y)\right) &= 0 \\
\int \frac{d}{dx}\left(G(x) + H(y)\right)\, dx &= \int 0\, dx \\
G(x) + H(y) - C &= 0,
\end{aligned}
$$

and so our solution is

$$G(x) + H(y) = C.$$

Now, if we are given an initial condition $y(x_0) = y_0$, we see that

$$G(x_0) + H(y_0) = C,$$

so

$$
\begin{aligned}
G(x) + H(y) &= G(x_0) + H(y_0) \\
G(x) - G(x_0) + H(y) - H(y_0) &= 0 \\
\int_{x_0}^{x} G'(s)\, ds + \int_{y_0}^{y} H'(s)\, ds &= 0.
\end{aligned}
$$

Now, recalling that $G'(x) = -g(x)$ and $H'(y) = \frac{1}{h(y)}$, we arrive at our solution

$$\int_{x_0}^{x} -g(s)\, ds + \int_{y_0}^{y} \frac{1}{h(s)}\, ds = 0,$$

which is an implicit representation of the solution. The final steps are to the perform the integrals and solve for y in terms of x.

Note: Often, determining an explicit formula is impossible, but you can use numerical methods to find approximate values of y for given values of x.

Now, let us solve the initial value problem

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}, \quad y(0) = 1.$$

We see that the differential equation is separable since

$$\frac{4x - x^3}{4 + y^3} = f(x)h(y),$$

where

$$f(x) = 4x - x^3, \quad h(y) = \frac{1}{4 + y^3}.$$

Now,

$$\begin{aligned}
\left(4+y^3\right)\frac{dy}{dx} &= 4x - x^3 \\
\frac{d}{dx}\int_1^y 4 + s^3\ ds &= 4x - x^3 \\
\int \frac{d}{dx}\int_1^y 4 + s^3\ ds\ dx &= \int 4x - x^3\ dx \\
\int_1^y 4 + s^3\ ds &= \frac{4x^2}{2} - \frac{x^4}{4} + C \\
4s + \frac{s^4}{4}\bigg|_1^y &= 2x^2 - \frac{x^4}{4} + C \\
4y + \frac{y^4}{4} - \left(4 + \frac{1}{4}\right) &= 2x^2 - \frac{x^4}{4} + C \\
16y + y^4 - 17 &= 8x^2 - x^4 + C.
\end{aligned}$$

Now, our initial condition informs us that $y(0) = 1$, so

$$\begin{aligned}
16 + 1 - 17 &= C \\
C &= 0.
\end{aligned}$$

Thus, our solution to the initial value problem is

$$16y + y^4 = 8x^2 - x^4 + 17.$$

We are left with an implicit representation of our solution which makes sense for all values of x, but for what values of x is this solution valid? Examining the differential equation, we see that the derivative blows up $\left(\left|\frac{dy}{dx}\right| \to \infty\right)$ when $4 + y^4 = 0$ or $y = (-4)^{\frac{1}{3}}$. The corresponding values of x are solutions to

$$16(-4)^{\frac{1}{3}} + (-4)^{\frac{4}{3}} = 8x^2 - x^4 + 17,$$

and so $x \approx \pm 3.35$. Therefore, the domain of validity of the differential equation is $x \in [-3.35, 3.35]$.

3 Modeling with First-Order Differential Equations

1. Construct the model
2. Analyze the model
3. Compare with experiment or observation