1 Modeling

Our motivation behind studying differential equations comes from their ability to model the physical phonomena that interest us. Our specializations vary, but the basic principles behind the construction of mathematical models follows the same ideas for each of us:

1. Construct the model
2. Analyze the model
3. Compare the model with our experiment or observation of interest

Constructing the model requies us to hypothesize about how our observations can be explained. Once we have built a model, we hope to analyze this model using the techniques we will study in this course. We can attempt to solve our governing equations (analytically or numerically), or we may attempt to gain qualitative insight about the predicted behavior. Once we understand how our model works, we then examine whether our model and observations align. If they do, then great! If they do not, we review the assumptions we made and see how we can improve our model to coincide with our experiment.

We will now construct a model using a first-order differential equation, and we will leave parameters in our differential equation, so that we may tune the model to our observations or test other hypothetical situations.

Lake Shasta in northern California is the state of California's largest reservoir, and it is formed by the Shasta Dam on the Sacramento river. We will create a model for the volume of water in Lake Shasta. Let $V(t)$ be the volume of water in the lake at time t, and we suppose that the volume of water in the lake is not constant. Let $r_1(t)$ be the inflow and $r_2(t)$ the outflow. Then,

$$\frac{dV}{dt} = r_1(t) - r_2(t).$$

How can we model $r_1(t)$?

- rain fall
- snow melt

We have not yet established a time scale, so let us choose the time scale to be years. Then,

$$r_1(t) = \cos(2\pi t)$$

gives us the desired shape for the inflow. Note that $t = 1$ is March of year 1, $t = 2$ is March of year 2, and so on. Next, we do not want negative flow, so let us add 1 to $r_1(t)$,

$$r_1(t) = \cos(2\pi t) + 1.$$

How should we model the outflow? For now, let us set $r_2(t) = c_1$. That is, there is a constant flow out of the lake. What are some potential problems we need to keep in mind when we choose c_1?

- If c_1 is too large, the lake could empty.
- If c_1 is too small, the lake could overflow.

What about initial conditions? The lake began storing water in 1944, so if we let $t = 0$ correspond to March 1944, then our initial condition is $V(0) = 0$. Thus, we have the initial value problem

$$\frac{dV}{dt} = \cos(2\pi t) + 1 - c_1, \quad V(0) = 0.$$

Now, we solve the initial value problem. We have

$$\int \frac{dV}{dt}\, dt = \int \cos(2\pi t) + 1 - c_1 \, dt$$
$$V(t) = \frac{1}{2\pi}\sin(2\pi t) + t - c_1 t + c_2.$$

Now, from the initial condition, we have

$$0 = V(0) = c_2.$$

Therefore, the solution to the initial value problem is

$$V(t) = \frac{1}{2\pi}\sin(2\pi t) + t - c_1 t.$$

We can find suitable values for c_1 by considering $V_{\max}$ and $V_{\min}$. That is, it may be undesirable to have the lake overflow or completely empty. You should check whether we can find c_1 so that the lake does not flow over the top of the dam.

2 Linear vs Nonlinear Differential Equations

Theorem. *Existence and uniqueness for first order linear differential equations. Let p and g be continuous functions on the interval $I : \alpha < t < \beta$ and $t_0 \in I$. Then, there is a unique function $y = \varphi(t)$ that satisfies the differential equation*

$$y' + p(t)y = g(t)$$

for each $t \in I$ and satsifies the initial condition $y(t_0) = y_0$.

Proof. We found that

$$\mu(t) = e^{\int p(t)\, dt}.$$

Since p is continuous for $\alpha < t < \beta$, μ is defined, differentiable, and nonzero. Next, μ and g are continuous, so μg is integrable and $\int \mu g\, dt$ is differentiable. Therefore,

$$y = \frac{1}{\mu(t)}\left(\int \mu(t)g(t)\, dt + C\right)$$

exists and is differentiable for $\alpha < t < \beta$. □

Theorem. *Existence and uniqueness for first order nonlinear differential equations. Let f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing (t_0, y_0). Then in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution to the initial value problem*

$$y' = f(t, y), \quad y(t_0) = y_0.$$

Note: The conditions in the theorem are sufficient, but not necessary. That is, we can make a weaker assumption on f. We can guarantee existence (but not uniqueness) by only assuming f is continuous.

Consider the initial value problem

$$y' = y^{\frac{1}{3}}, \quad y(0) = 0$$

for $t \geq 0$. We have that $f(t, y) = y^{\frac{1}{3}}$, so $\frac{\partial f}{\partial y} = \frac{1}{3}y^{-\frac{2}{3}}$. Notice that f is continuous everywhere, but $\frac{\partial f}{\partial y}$ does not exist when $y = 0$. Therefore, the previous theorem does not apply. Since f is continuous, a solution exists, but it is not necessarily unique.

Let us solve the initial value problem, and see the implication of not having a unique solution. Solving the differential equation, we have

$$\begin{aligned} y^{-\frac{1}{3}}y' &= 1 \\ \frac{3}{2}y^{\frac{2}{3}} &= t + c \\ y &= \left(\frac{2}{3}(t + c)\right)^{\frac{3}{2}}. \end{aligned}$$

Now, using the initial condition gives

$$0 = y(0) = \left(\frac{2}{3}c\right)^{\frac{3}{2}},$$

so $c = 0$. Therefore,

$$\varphi_1(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}}, \quad t \geq 0,$$

is a solution to the initial value problem. But also,

$$\varphi_2(t) = -\left(\frac{2}{3}t\right)^{\frac{3}{2}}, \quad t \geq 0$$

is a solution to the initial value problem. Notice that for any value of t_0,

$$y = \begin{cases} 0 & 0 \leq t < t_0 \\ \pm\left(\frac{2}{3}(t - t_0)\right)^{\frac{3}{2}}, & t \geq t_0 \end{cases}$$

is a solution to the initial value problem.

2.1 Interval of Existence

- Linear equations: The solution exists in the interval about $t = t_0$ in which p and g are continuous.
- Nonlinear equations: $y = \varphi(t)$ exists as long as $(t, \varphi(t))$ remains in the hypothesis region. Note that $\varphi(t)$ is not known ahead of time.

2.2 General Solution

- Linear equations: We can obtain all possible solutions by specifying the constant of integration.
- Nonlinear equations: There may be solutions that cannot be obtained by giving values to the constant.

2.3 Implicit Solutions

- Linear equations: $y = \varphi(t)$ (explicit)
- Nonlinear equations: $F(t, y) = 0$ (implicit)

3 Autonomous Differential Equations and Population Dynamics

A first order ODE in which the independent variable does not appear explicitly is called autonomous, and it is of the form

$$\frac{dy}{dt} = f(y).$$

This equation is separable, and we will examine this equation in the context of population dynamics.

3.1 Exponential Growth

Suppose the rate of change of a population is proportional to the current population. Then,

$$\frac{dy}{dt} = ry,$$

where r is the growth rate, and y is the population. If we specify that $y(0) = y_0$, then our solution to the initial value problem is $y = y_0 e^{rt}$. Then, we see that

- if $r > 0$, then $y \to \infty$ as $t \to \infty$
- if $r < 0$, then $y \to 0$ as $t \to \infty$
- $r = 0$, then $y = y_0$ for all t

3.2 Logistic Growth

Suppose the growth rate depends on the population. Then we have

$$\frac{dy}{dt} = h(y)y.$$

Consider $h(y) = r - ay$, where $a > 0$. Then,

$$\frac{dy}{dt} = (r - ay)y,$$

or

$$\frac{dy}{dt} = r\left(1 - \frac{y}{k}\right)y$$

where $k = \frac{r}{a}$. The parameter r is called the intrinsic growth rate, and it represents the rate of growth when there are no external influences. This differential equation is called the logistic equation.

3.3 Qualitative Analysis

Identifying equilibrium solutions gives us insight into the behavior of solutions. Equilibrium solutions are solution for which $\frac{dy}{dt} = 0$ for all t. so we may find these equilibrium solutions by determining the values of y for wrhich $f(y) = 0$. Also, $\frac{dy}{dt} = f(y) = 0$, so we are looking for the critical points of y.