1 Autonomous Differential Equations and Population Dynamics

A first order ODE in which the independent variable does not appear explicitly is called autonomous, and it is of the form

$$\frac{dy}{dt} = f(y).$$

This equation is separable, and we will examine this equation in the context of population dynamics.

1.1 Exponential Growth

Suppose the rate of change of a population is proportional to the current population. Then,

$$\frac{dy}{dt} = ry,$$

where r is the growth rate, and y is the population. If we specify that $y(0) = y_0$, then our solution to the initial value problem is $y = y_0 e^{rt}$. Now, let us examine the behavior of solutions.

- $r > 0$
  - If $y > 0$, then $\frac{dy}{dt} > 0$.
  - If $y < 0$, then $\frac{dy}{dt} < 0$.
  - If $y = 0$, then $\frac{dy}{dt} = 0$.
- $r < 0$
  - If $y > 0$, then $\frac{dy}{dt} < 0$
  - If $y < 0$, then $\frac{dy}{dt} > 0$
  - If $y = 0$, then $\frac{dy}{dt} = 0$
- $r = 0$ then $y = y_0$ for all t

From our plots of the direction field, we see that the solution trajectories exhibit different behaviors. When $r > 0$, solution trajectories move away from the equilibrium solution, but when $r < 0$, solution trajectories tend towards the equilibrium solution.

1.2 Logistic Growth

Now suppose that the growth rate depends on the population. Then we have

$$\frac{dy}{dt} = h(y)y.$$

Consider $h(y) = r - ay$, where $a > 0$. Then,

$$\frac{dy}{dt} = (r - ay)y,$$

or

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

where $K = \frac{r}{a}$. The parameter r is called the intrinsic growth rate, and it represents the rate of growth when there are no external influences. This differential equation is called the logistic equation.

1.3 Qualitative Analysis

Identifying equilibrium solutions gives us insight into the behavior of solutions. Equilibrium solutions are solutions for which $\frac{dy}{dt} = 0$ for all t. We may find these equilibrium solutions by determining the values of y for which $f(y) = 0$. Also, $\frac{dy}{dt} = f(y) = 0$, so we are looking for the critical points of y.

Now, let us analyze the logistic equation. We see that $f(y) = r\left(1 - \frac{y}{K}\right)y$, so we want to find the values of y such that

$$0 = r\left(1 - \frac{y}{K}\right)y.$$

We see that $y = K$ and $y = 0$ are two such solutions, so these are our two equilibrium solutions. Now, let us analyze the behavior of solutions when our initial condition is not an equilibrium solution. If $y \in (0, k)$, $\frac{y}{k} < 1$, so $1 - \frac{y}{k} > 0$. Therefore $\frac{dy}{dt} = f(y) > 0$. For $y \in (k, \infty)$, $\frac{y}{k} > 1$, so $1 - \frac{y}{k} < 0$. Therefore $\frac{dy}{dt} = f(y) < 0$. The y-axis is called the phase line, and note that previously, we plotted y vs t. We can determine the concavity of curves using the observation that

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\frac{dy}{dt} = \frac{d}{dt}f(y) = f'(y)\frac{dy}{dt} = f'(y)f(y).$$

Now, from our plot of the phase line, we see that solutions tend to K as $t \to \infty$, and K is called the saturation level or carrying capacity. The equilibrium $y = K$ is called a stable equilibrium, and $y = 0$ is called an unstable equilibrium.

1.4 Critical Threshold

Consider

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)y.$$

T is called the threshold level. If $y > T$, then we have growth. If $y < T$, then we have decay.

1.5 Logistic Growth with a Threshold

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{k}\right)y, \quad T < k.$$

In this model, we now have both a threshold and a saturation level.

2 Numerical Approxmations: Euler's Method

We have seen a few examples of first order differential equations that we can solve symbolically, but more often than not, we cannot solve the differential equation "by hand." We know that the IVP

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

has a unique solution $y = \varphi(t)$ if f and $\frac{\partial f}{\partial y}$ are continuous, so how can we "see" the solution?

One method is to draw a direction field and sketch some solution curves. This gives us good qualitative results, but we may want more concrete quantitative results. When we draw solution curves, what are we doing? This is exactly the idea behind the tangent line method or Euler's method!

Consider the IVP

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

The solution passes through (t_0, y_0), and the slope at (t_0, y_0) is $f(t_0, y_0)$. Therefore, the tangent line to the solution at (t_0, y_0) is

$$y = y_0 + f(t_0, y_0)(t - t_0),$$

and as long as we stay close to $t = t_0$, this is a good approximation.

Say we want a tangent line approximation at $t = t_1$. Let's use the tangent line at (t_0, y_0) to approximate y_1, so

$$y_1 = y_0 + f(t_0, y_0)(t_1 - t_0).$$

Next, we want to find (t_2, y_2), so using the tangent line at (t_1, y_1), we have

$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1),$$

and so

$$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n).$$

This procedure produces a sequence $y_0, y_1, \dots$ at $t_0, t_1, \dots$ which gives an approximation to the solution.

If we assume a uniform step size h between $t_0, t_1, t_2, \dots$, then $t_{n+1} = t_n + h$, so

$$y_{n+1} = y_n + hf(t_n, y_n).$$

Euler's method is a sequence of tedious computations, so rather than doing this method by hand, we will ask a computer to do the computations for us.

The numerical method above is called (forward/explicit) Euler's method. There is also what's called backward/implicit Euler's method. Instead of evaluating $f(t, y)$ at the current value (t_n, y_n), we evaluate f at (t_{n+1}, y_{n+1}). Then, our method becomes

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}).$$

For a specific funciton f, we hope to rearrange the equation so that we may explicitly write y_{n+1} in terms of y_n, h, and t_{n+1}.

Set up the Euler iteration scheme for the IVP

$$\frac{dy}{dt} = t^2 + 5y, \quad y(0) = 1.$$

Explicit:

$$\begin{aligned} y_{n+1} &= y_n + hf(t_n, y_n) \\ &= y_n + h(t_n^2 + 5y_n) \\ &= y_n(1 + 5h) + ht_n^2. \end{aligned}$$

Implicit:

$$\begin{aligned} y_{n+1} &= y_n + hf(t_{n+1}, y_{n+1}) \\ y_{n+1} &= y_n + h(t_{n+1}^2 + 5y_{n+1}) \\ y_{n+1}(1 - 5h) &= y_n + ht_{n+1}^2 \\ y_{n+1} &= \frac{y_n + ht_{n+1}^2}{1 - 5h} \\ y_{n+1} &= \frac{y_n + h(t_n + h)^2}{1 - 5h}. \end{aligned}$$