1 First-Order Difference Equations

So far, we have considered continuous models, but now we explore discrete models which will lead us to difference equations. A first-order difference equation is of the form

$$y_{n+1} = f(n, y_n), \quad n = 0, 1, 2, \ldots.$$

The difference equation is called first-order, because f is a function of y_n and not any other previous values such as $y_{n-1}, y_{n-2}, \ldots$. The difference equation is linear if f is a linear function of y_n; otherwise it is nonlinear. A solution is a sequence of numbers $y_0, y_1, \ldots$ that satisfies the equation for each n. The initial condition $y_0 = \alpha$ specifies the first term of the solution sequence, and we may compute the terms that follow.

1.1 Equilibrium Solutions

Suppose

$$y_{n+1} = f(y_n), \quad n = 0, 1, 2, \ldots,$$

where f only depends on y_n. Then,

$$\begin{aligned} y_1 &= f(y_0) \\ y_2 &= f(y_1) = f(f(y_0)) \\ y_3 &= f(y_2) = f(f(f(y_0))) = f^3(y_0), \end{aligned}$$

and so

$$y_n = f^n(y_0).$$

We may want to understand the behavior of y_n as $n \to \infty$, and equilibrium solutions will give us a picture of how solutions behave. If y_n has the same value for all n, y_n is called an equilibrium solution, and such a solution satisfies the equation

$$y_n = f(y_n).$$

1.2 Linear Equations

Consider the difference equation

$$y_{n+1} = \rho y_n + b_n, \quad n = 0, 1, 2, \ldots.$$

First, let's solve the equation in terms of the initial value y_0. To do this, let's compute $y_1, y_2, \ldots$, and see if we spot a pattern developing. We have

$$y_1 = \rho y_0 + b_0,$$

$$\begin{aligned} y_2 &= \rho y_1 + b_1 \\ &= \rho\left(\rho y_0 + b_0\right) + b_1 \\ &= \rho^2 y_0 + \rho b_0 + b_1, \end{aligned}$$

and

$$\begin{aligned} y_3 &= \rho y_2 + b_2 \\ &= \rho\left(\rho^2 y_0 + \rho b_0 + b_1\right) + b_2 \\ &= \rho^3 y_0 + \rho^2 b_0 + \rho b_1 + b_2. \end{aligned}$$

After comparing y_1, y_2, and y_3, we see that

$$\begin{aligned} y_n &= \rho^n y_0 + \rho^{n-1} b_0 + \cdots + \rho b_{n-2} + b_{n-1} \\ &= \rho^n y_0 + \sum_{j=0}^{n-1} \rho^{n-1-j} b_j. \end{aligned}$$

Now, let us suppose that $b_n = b \neq 0$ for all n. Then,

$$\begin{aligned} y_n &= \rho^n y_0 + b\sum_{j=0}^{n-1} \rho^{n-1-j} \\ &= \rho^n y_0 + b\frac{1-\rho^n}{1-\rho} \\ &= \rho^n \left(y_0 - \frac{b}{1-\rho}\right) + \frac{b}{1-\rho}. \end{aligned}$$

Now, let's determine the long-time behavior of y_n.

1. If $|\rho| < 1$, then $y_n \to \frac{b}{1-\rho}$ as $n \to \infty$.
2. Suppose $|\rho| > 1$.
   (a) If $y_0 = \frac{b}{1-\rho}$, then $y_n = \frac{b}{1-\rho}$ for all n.
   (b) If $y_0 \neq \frac{b}{1-\rho}$, then there is no (finite) limit.
3. If $\rho = -1$, there is no limit.
4. If $\rho = 1$, we need to go back to the difference equation,

$$\begin{aligned} y_{n+1} &= \rho y_n + b \\ &= y_n + b \end{aligned}$$

   and so $y_n \to \infty$ as $n \to \infty$.

1.3 Nonlinear Equations

Consider the logistic difference equation

$$y_{n+1} = \rho y_n \left(1 - \frac{y_n}{k}\right)$$

or

$$u_{n+1} = \rho u_n \left(1 - u_n\right),$$

where we used the change of variable

$$u_n = \frac{y_n}{k}.$$

Now let's find equilibrium solutions,

$$\begin{aligned} u_n &= \rho u_n \left(1 - u_n\right) \\ \rho u_n^2 + u_n(1-\rho) &= 0 \\ u_n \left(\rho u_n + 1 - \rho\right) &= 0, \end{aligned}$$

and so $u_n = 0$ and $u_n = \frac{\rho-1}{\rho}$ are our equilibrium solutions. Now, let us draw the associated cobweb (stairstep) diagram.

2 Second-Order Linear Differential Equations

After exploring first-order differential equations, we now turn our focus to second-order linear differential equations. Consider the following differential equation,

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right).$$

This is a second-order differential equation, and it is linear if

$$f\left(t, y, \frac{dy}{dt}\right) = g(t) - p(t)\frac{dy}{dt} - q(t)y,$$

where g, p, and q are given functions of t. Our general second-order linear differential equation is

$$y'' + p(t)y' + q(t)y = g(t).$$

The initial value problem now requires us to specify

$$y(t_0) = y_0$$

and

$$y'(t_0) = y_0'.$$

That is, we now require specifing an initial point (t_0, y_0) and an initial slope y_0'. The second-order differential equation is called homogeneous if $g(t) = 0$ for all t. Otherwise, it is called nonhomogeneous. Solving the nonhomogeneous problem requires us to first solve the homogeneous problem, so we now examine the homogeneous case.

3 Homogeneous Differential Equations with Constant Coefficients

Consider

$$ay'' + by' + cy = 0,$$

where a, b, and c are given constants. Recall that we have already seen how to solve the differential equation! Solving the second-order linear constant coefficient homogeneous differential equation relies on two key ideas: a good ansatz and linearity.

3.1 Idea I: Ansatz and the Characteristic Equation

Suppose $y = e^{rt}$ is a solution to the differential equation. Then

$$\begin{aligned} ay'' + by' + cy &= a\left(e^{rt}\right)'' + b\left(e^{rt}\right)' + ce^{rt} \\ &= ar^2 e^{rt} + bre^{rt} + ce^{rt} \\ &= e^{rt}\left(ar^2 + br + c\right), \end{aligned}$$

and so we want

$$e^{rt}\left(ar^2 + br + c\right) = 0.$$

Now, $e^{rt} \neq 0$, so we require

$$ar^2 + br + c = 0.$$

This equation is called the characteristic equation of the differential equation and has two roots, r_1 and r_2. Depending on the constants a, b, and c, we may encounter roots which are real and distinct, real and repeated, or complex conjugates. For now, let us assume r_1 and r_2 are distinct real roots. Then,

$$y_1 = e^{r_1 t}$$

and

$$y_2 = e^{r_2 t}$$

are both solutions to the differential equation.

3.2 Idea II: Linearity

If y_1 and y_2 are solutions of the differential equation, then $y = c_1 y_1 + c_2 y_2$ solves the differential equation. Why? We see that

$$\begin{aligned} ay'' + by' + cy &= a\left(c_1 y_1 + c_2 y_2\right)'' + b\left(c_1 y_1 + c_2 y_2\right)' + c\left(c_1 y_1 + c_2 y_2\right) \\ &= ac_1 y_1'' + bc_1 y_1' + cc_1 y_1 + ac_2 y_2'' + bc_2 y_2' + cc_2 y_2 \\ &= c_1\left(ar_1^2 + br_1 + c\right)e^{r_1 t} + c_2\left(ar_2^2 + br_2 + c\right)e^{r_2 t} \\ &= 0, \end{aligned}$$

since r_1 and r_2 are roots of $p(r) = ar^2 + br + c$.

3.3 The Initial Value Problem

We have confirmed that

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

solves the differential equation, and we call this the general solution. We obtain a specific solution to the initial value problem by determining the constants c_1 and c_2 using the initial values

$$y(t_0) = y_0$$

and

$$y'(t_0) = y_0'.$$

Consider the initial value problem

$$4y'' - 8y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{2}.$$

First, the characteristic equation is

$$p(r) = 4r^2 - 8r + 3.$$

Next, we find the roots of $p(r)$,

$$\begin{aligned} 4r^2 - 8r + 3 &= 0 \\ (2r-1)(2r-3) &= 0, \end{aligned}$$

and so $r_1 = \frac{3}{2}$ and $r_2 = \frac{1}{2}$. Thus, our general solution is

$$y = c_1 e^{\frac{3}{2}t} + c_2 e^{\frac{1}{2}t}.$$

Now, we determine c_1 and c_2 using the initial conditions. We have

$$\begin{aligned} 2 &= y(0) \\ &= c_1 + c_2, \end{aligned}$$

so

$$c_1 = 2 - c_2.$$

We now have that

$$y = (2 - c_2)e^{\frac{3}{2}t} + c_2 e^{\frac{1}{2}t},$$

so

$$y' = \frac{3}{2}(2 - c_2)e^{\frac{3}{2}t} + \frac{1}{2}c_2 e^{\frac{1}{2}t}.$$

Using the second initial condition, we have

$$\begin{aligned} \frac{1}{2} &= y'(0) \\ \frac{1}{2} &= \frac{3}{2}(2 - c_2) + \frac{1}{2}c_2 \\ 1 &= 3(2 - c_2) + c_2 \\ -5 &= -2c_2 \\ c_2 &= \frac{5}{2}, \end{aligned}$$

and so

$$\begin{aligned} c_1 &= 2 - c_2 \\ &= 2 - \frac{5}{2} \\ &= -\frac{1}{2}. \end{aligned}$$

Thus, our solution to the initial value problem is

$$y = -\frac{1}{2}e^{\frac{3}{2}t} + \frac{5}{2}e^{\frac{1}{2}t}.$$

3.4 Long-time Behavior

Our general solution to the second-order linear constant coefficient differential equation is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

What happens as $t \to \infty$? We see that the behavior of the solution as $t \to \infty$ depends on the roots of the characteristic equation. We will return to this question after figuring out how our general solution presents itself when we come across complex and real repeated roots.