# 1 Solutions of the Homogeneous Equations and the Wronskian

Let $p$ and $q$ be continuous functions on an open interval $I$. Consider the differential operator $L$ defined by

$$L[\varphi] = \varphi'' + p\varphi' + q\varphi,$$

where $\varphi$ is some function. Here, $L$ operates on $\varphi$. That is to say, $\varphi$ acts as the input to the operator $L$. The value of $L[\varphi]$ at a point $t$ is

$$L[\varphi](t) = \varphi''(t) + p(t)\varphi'(t) + q(t)\varphi(t).$$

We will examine the second-order linear differential equation

$$L[\varphi](t) = 0,$$

with initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0.$$

We want to know whether the initial value problem has a solution, whether it has more than one solution and how solutions are structured and formed.

**Theorem.** *(Existence and Uniqueness) Let $p$, $g$, and $q$ be continuous in an open interval $I$ containing $t_0$. Then the initial value problem*

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0',$$

*has a unique solution $y = \varphi(t)$ in the interval $I$.*

**Theorem.** *If $y_1$ and $y_2$ are solutions to the differential equation*

$$L[y] = y'' + p(t)y' + q(t) = 0,$$

*then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any $c_1$ and $c_2$.*

The proof of the previous theorem follows from the linearity of $L$. Now, when can $c_1$ and $c_2$ be chosen to satisfy the initial conditions? We require

$$c_1y_1(t_0) + c_2y_2(t_0) = y_0$$
$$c_1y_1'(t_0) + c_2y_2'(t_0) = y_0',$$

or in matrix form

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}.$$

When does this system have a unique solution? Let

$$W(t_0) = \det\left(\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}\right).$$

If $W(t_0) \neq 0$, then we are guaranteed a unique solution $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, and $W$ is called the Wronskian of the solutions $y_1$ and $y_2$.

**Theorem.** *Suppose $y_1$ and $y_2$ are two solutions of*

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

*It is possible to choose $c_1$ and $c_2$ so that*

$$y = c_1y_1(t) + c_2y_1(t)$$

*satisfies the differential equation and initial conditions if and only if the Wronskian*

$$W[y_1, y_2] = y_1y_2' - y_1'y_2$$

*is not zero at $t_0$.*

Consider the differential equation

$$y'' + 5y' + 6y = 0.$$

Find the Wronskian of $y_1$ and $y_2$. The characteristic equation is

$$\begin{aligned} p(r) &= r^2 + 5r + 6 \\ &= (r+3)(r+2), \end{aligned}$$

so our roots are $r_1 = -3$ and $r_2 = -2$. Therefore, our solutions are

$$y_1 = e^{-2t}$$

and

$$y_2 = e^{-3t}.$$

Now,

$$\begin{aligned} W[y_1, y_2](t) &= \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} \\ &= -3e^{-5t} + 2e^{-5t} \\ &= -e^{-5t}. \end{aligned}$$

We see that $W[y_1, y_2](t) \neq 0$ for all $t$, so any initial condition can be specified at any initial time $t$. The expression

$$y = c_1 y_1(t) + c_2 y_2(t)$$

is called the general solution of $L[y] = 0$, and $y_1$ and $y_2$ are said to form a fundamental set of solutions if their Wronskian is nonzero for all $t$. Notice that we have a nonzero Wronskian if and only if $y_1$ and $y_2$ are linearly independent.

Show that $y_1 = t^{\frac{1}{2}}$ and $y_2 = \frac{1}{t}$ form a fundamental set of solutions of

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0.$$

First, we need to show that $y_1$ and $y_2$ satisfy the differential equation. Using $y_1$ in the differential equation, we have

$$\begin{aligned} 2t^2 y_1'' + 3ty_1' - y_1 &= 2t^2 (t^{\frac{1}{2}})'' + 3t(t^{\frac{1}{2}})' - t^{\frac{1}{2}} \\ &= 2t^2 \left(\frac{1}{2} t^{-\frac{1}{2}}\right)' + 3t \frac{1}{2} t^{-\frac{1}{2}} - t^{\frac{1}{2}} \\ &= 2t^2 \left(-\frac{1}{4}\right) t^{-\frac{3}{2}} + \frac{3}{2} t^{\frac{1}{2}} - t^{\frac{1}{2}} \\ &= -\frac{1}{2} t^{\frac{1}{2}} + \frac{1}{2} t^{\frac{1}{2}} \\ &= 0, \end{aligned}$$

so $y_1$ is a solution. Next, we use $y_2$ in the differential equation,

$$\begin{aligned} 2t^2 y_2'' + 3ty_2' - y_2 &= 2t^2 \left(\frac{1}{t}\right)'' + 3t \left(\frac{1}{t}\right)' - \frac{1}{t} \\ &= 2t^2 \left(-\frac{1}{t^2}\right)' + 3t \left(-\frac{1}{t^2}\right) - \frac{1}{t} \\ &= 2t^2 \frac{2}{t^3} - \frac{3}{t} - \frac{1}{t} \\ &= \frac{4}{t} - \frac{4}{t} \\ &= 0, \end{aligned}$$

so $y_2$ is a solution. Now, we compute the Wronskian

$$\begin{aligned} W[y_1, y_2](t) &= \begin{vmatrix} t^{\frac{1}{2}} & \frac{1}{t} \\ \frac{1}{2}t^{-\frac{1}{2}} & -\frac{1}{t^2} \end{vmatrix} \\ &= t^{\frac{1}{2}}\left(-t^{-2}\right) - \frac{1}{2}t^{-\frac{1}{2}}t^{-1} \\ &= -t^{-\frac{3}{2}} - \frac{1}{2}t^{-\frac{3}{2}} \\ &= -\frac{3}{2}t^{-\frac{3}{2}}. \end{aligned}$$

We see that $W[y_1, y_2](t) \neq 0$ for $t > 0$, so $y_1$ and $y_2$ form a fundamental set of solutions. Thus, the general solution to the differential equation is

$$y(t) = c_1 t^{\frac{1}{2}} + c_2 \frac{1}{t}.$$

**Theorem.** *If $y = u(t) + iv(t)$ is a complex-valued solution of $L[\varphi] = 0$, then its real part $u$ and imaginary part $v$ are also solutions.*

The proof follows form the linearity of $L$, that $L[y] = 0$, and the fact that if $a + ib = 0$, then $a = 0$ and $b = 0$.

**Theorem.** *(Abel's Theorem) If $y_1$ and $y_2$ are solutions of*

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

*where $p$ and $q$ are continuous on $I$, then the Wronskain is*

$$W[y_1, y_2](t) = Ce^{-\int p(t)\,dt},$$

*where $C$ is a constant depending on $y_1$ and $y_2$, but not on $t$. $W$ is either zero for all $t$ in $I$ or never zero in $I$.*

# 2 Complex Roots of the Characteristic Equation

Our second-order constant coefficient homogeneous differential equation is

$$ay'' + by' + cy = 0,$$

where $a$, $b$, and $c$ are constants. The characteristic equation is

$$p(r) = ar^2 + br + c,$$

so

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

What happens when $b^2 - 4ac < 0$? We have roots $r_1$ and $r_2$ which are complex conjugates of each other

$$r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu,$$

where $\lambda$ and $\mu$ are real numbers. Our solutions are

$$y_1 = e^{(\lambda + i\mu)t}$$

and

$$y_2 = e^{(\lambda - i\mu)t}.$$

What does Euler's number raised to a complex power even mean?

## 2.1 Euler's Formula

Recall the Taylor (Maclaurin) series for $e^t$ about $t = 0$,

$$\begin{aligned} e^t &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \\ &= 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \cdots . \end{aligned}$$

Now,

$$\begin{aligned} e^{it} &= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \\ &= 1 + it - \frac{t^2}{2!} - \frac{it^3}{3!} + \frac{t^4}{4!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \\ &= \cos(t) + i\sin(t). \end{aligned}$$

Thus,

$$e^{it} = \cos(t) + i\sin(t),$$

and this is called Euler's formula. Now,

$$\begin{aligned} y_1 &= e^{(\lambda + i\mu)t} \\ &= e^{\lambda t} e^{i\mu t} \\ &= e^{\lambda t} \left(\cos(\mu t) + i\sin(\mu t)\right) \end{aligned}$$

and

$$\begin{aligned} y_2 &= e^{(\lambda - i\mu)t} \\ &= e^{\lambda t} e^{-i\mu t} \\ &= e^{\lambda t} \left(\cos(\mu t) - i\sin(\mu t)\right), \end{aligned}$$

where we recalled that sine is an odd function

$$\sin(-t) = -\sin(t),$$

and cosine is an even function

$$\cos(-t) = \cos(t).$$

## 2.2 Complex Roots: The General Solution

Recall that if $y = u + iv$ is a solution, then $u$ and $v$ are each solutions, so $u = e^{\lambda t}\cos(\mu t)$ and $v = e^{\lambda t}\sin(\mu t)$ are our solutions. The Wronskian is

$$\begin{aligned} W[u,v] &= \begin{vmatrix} e^{\lambda t}\cos(\mu t) & e^{\lambda t}\sin(\mu t) \\ \lambda e^{\lambda t}\cos(\mu t) - \mu e^{\lambda t}\sin(\mu t) & \lambda e^{\lambda t}\sin(\mu t) + \mu e^{\lambda t}\cos(\mu t) \end{vmatrix} \\ &= \lambda e^{2\lambda t}\cos(\mu t)\sin(\mu t) + \mu e^{2\lambda t}\cos^2(\mu t) - \lambda e^{2\lambda t}\sin(\mu t)\cos(\mu t) + \mu e^{2\lambda t}\sin^2(\mu t) \\ &= \mu e^{2\lambda t}\left(\cos^2(\mu t) + \sin^2(\mu t)\right) \\ &= \mu e^{2\lambda t}, \end{aligned}$$

so, $W[u,v] \neq 0$ if $\mu \neq 0$. Therefore, $u$ and $v$ form a fundamental set of solutions if $\mu \neq 0$. Notice that if $\mu = 0$, then $r_1 = \lambda$ and $r_2 = \lambda$, so we have real repeated roots. We will return to this case later, but when the characteristic equation has complex roots, we see that the general solution of the differential equation is

$$y(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t).$$

Consider the following initial value problem

$$y'' + y' + 9.25y = 0, \quad y(0) = 2, \quad y'(0) = 0.$$

The characteristic equation is

$$p(r) = r^2 + r + 9.25,$$

and so

$$\begin{aligned} r &= \frac{-1 \pm \sqrt{1-37}}{2} \\ &= \frac{-1 \pm 6i}{2} \\ &= -\frac{1}{2} \pm 3i. \end{aligned}$$

Thus, our general solution is

$$y = c_1 e^{-\frac{1}{2}t}\cos(3t) + c_2 e^{-\frac{1}{2}t}\sin(3t).$$

Now, we use the initial conditions to determine $c_1$ and $c_2$. We have

$$\begin{aligned} 2 &= y(0) \\ &= c_1, \end{aligned}$$

so

$$y = 2e^{-\frac{1}{2}t}\cos(3t) + c_2 e^{-\frac{1}{2}t}\sin(3t).$$

Next,

$$y' = -e^{-\frac{1}{2}t}\cos(3t) - 6e^{-\frac{1}{2}t}\sin(3t) - \frac{c_2}{2}e^{-\frac{1}{2}t}\sin(3t) + 3c_2 e^{-\frac{1}{2}t}\cos(3t),$$

so

$$\begin{aligned} 0 &= y'(0) \\ 0 &= -1 + 3c_2 \\ \frac{1}{3} &= c_2. \end{aligned}$$

Thus, the solution to the initial value problem is

$$y = 2e^{-\frac{1}{2}t}\cos(3t) + \frac{1}{3}e^{-\frac{1}{2}t}\sin(3t).$$

Notice that the solution oscillates with a decaying amplitude, and $\lim_{t\to\infty} y(t) = 0$.