1 Repeated Roots

Our roots of the characteristic equation are

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

and we now consider the case where $b^2 - 4ac = 0$, or $r = -\frac{b}{2a}$. We have real repeated roots, so we only have the solution

$$y_1 = e^{-\frac{b}{2a}t}.$$

What about the second solution? We know that if y_1 is a solution, then $y_2 = cy_1$ is also a solution, but we need y_1 and y_2 to be linearly indepedent so that they form a fundamental set of solutions. Let us extend this idea by considering $y_2 = v(t)y_1$ where $v(t)$ is a to-be-determined function. For y_2 to be a solution, it needs to satisfy the differential equation. We have

$$y_2' = v'(t)y_1 + v(t)y_1'$$

and

$$y_2'' = v''(t)y_1 + 2v'(t)y_1' + v(t)y_1''.$$

Now, using our results in the differential equation, we have

$$\begin{aligned} ay_2'' + by_2' + cy_2 &= a\left(v''(t)y_1 + 2v'(t)y_1' + v(t)y_1''\right) + b\left(v'(t)y_1 + v(t)y_1'\right) + cv(t)y_1 \\ &= av''(t)y_1 + v'(t)(2ay_1' + by_1) + v(t)(ay_1'' + by_1' + cy_1) \\ &= av''(t)y_1 + v'(t)(2ay_1' + by_1), \end{aligned}$$

where the last line follows from y_1 being a solution to the differential equation. Now, we know $y_1 = e^{-\frac{b}{2a}t}$, so

$$\begin{aligned} av''(t)y_1 + v'(t)(2ay_1' + by_1) &= av''(t)e^{-\frac{b}{2a}t} + v'(t)\left(2a\left(-\frac{b}{2a}\right)e^{-\frac{b}{2a}t} + be^{-\frac{b}{2a}t}\right) \\ &= ae^{-\frac{b}{2a}t}v''(t). \end{aligned}$$

We require

$$ae^{-\frac{b}{2a}t}v''(t) = 0$$

for y_2 to be a solution, so we see that we must have that $v''(t) = 0$. Therefore, $v(t) = c_3t + c_4$, so

$$y_2 = \left(c_3t + c_4\right)y_1.$$

Thus, our general solution is of the form

$$y = c_1y_1 + c_2ty_1.$$

2 Reduction of Order

Consider the second-order differential equation

$$y'' + p(t)y' + q(t)y = 0,$$

and suppose we know $y_1(t)$ is a solution. Following the same idea as in the previous section, let $y_2 = v(t)y_1(t)$. Then

$$y_2' = v'y_1 + vy_1'$$

and

$$\begin{aligned} y_2'' &= v''y_1 + v'y_1' + v'y_1' + vy_1'' \\ &= v''y_1 + 2v'y_1' + vy_1''. \end{aligned}$$

Now, using these results in the differential equation, we get

$$\begin{aligned} y'' + p(t)y' + q(t)y &= v''y_1 + 2v'y_1' + vy_1'' + p(t)\left(v'y_1 + vy_1'\right) + q(t)v(t)y_1(t) \\ &= v''y_1 + v'\left(2y_1' + p(t)y_1\right) + v\left(y_1'' + p(t)y_1' + q(t)y_1\right) \\ &= v''y_1 + v'\left(2y_1' + p(t)y_1\right), \end{aligned}$$

where the last line follow from y_1 being a solution. We now arrive at a differential equaiton for v,

$$y_1v'' + (2y_1' + py_1)v' = 0.$$

Note that y_1 is known, so the second-order differential equation for v is really a first-order differential equation for v'. Let $w = v'$, so that we get

$$y_1w' + (2y_1' + py_1)w = 0.$$

Notice that this differential equation is separable, so we can determine w. Once we acquire w, we can find v by integrating w.

Suppose $y_1 = t^{-1}$ is a solution of

$$2t^2y'' + 3ty' - y = 0, \quad t > 0.$$

Find a fundamental set of solutions. Let $y_2 = v(t)t^{-1}$. Then,

$$y_2' = v't^{-1} - vt^{-2},$$

and

$$\begin{aligned} y_2'' &= v''t^{-1} - v't^{-2} - v't^{-2} + 2vt^{-3} \\ &= v''t^{-1} - 2v't^{-2} + 2vt^{-3}. \end{aligned}$$

Now, plugging into the differential equation,

$$\begin{aligned} 2t^2y_2'' + 3ty_2' - y_2 &= 2t^2\left(v''t^{-1} - 2v't^{-2} + 2vt^{-3}\right) + 3t\left(v't^{-1} - vt^{-2}\right) - vt^{-1} \\ &= 2v''t - 4v' + 4vt^{-1} + 3v' - 3vt^{-1} - vt^{-1} \\ &= 2v''t - v', \end{aligned}$$

so our differential equation for v' is

$$2v''t - v' = 0.$$

Let $w = v'$. Then,

$$\begin{aligned} 2w't - w &= 0 \\ w' &= \frac{w}{2t} \\ \frac{w'}{w} &= \frac{1}{2t} \\ \frac{d}{dt}(\ln w) &= \frac{1}{2t} \\ \int \frac{d}{dt}(\ln w)\ dt &= \int \frac{1}{2t}\ dt \\ \ln w &= \frac{1}{2}\ln t + c_3 \\ w &= c_4 e^{\frac{1}{2}\ln t} \\ w &= c_4 e^{\ln t^{\frac{1}{2}}} \\ w &= c_4 t^{\frac{1}{2}}. \end{aligned}$$

Now, $w = v'$, so

$$\begin{aligned} v' &= c_4 t^{\frac{1}{2}} \\ \int v' \; dt &= \int c_4 t^{\frac{1}{2}} \; dt \\ v(t) &= c_5 t^{\frac{3}{2}} + c_6. \end{aligned}$$

Therefore,

$$\begin{aligned} y_2 &= v y_1 \\ &= \left(c_5 t^{\frac{3}{2}} + c_6 \right) t^{-1} \\ &= c_5 t^{\frac{1}{2}} + c_6 t^{-1}, \end{aligned}$$

so our general solution is

$$y = c_1 t^{-1} + c_2 t^{\frac{1}{2}}.$$

Finally, let us check that we indeed have a fundamental set of solutions. The Wronskian is

$$\begin{aligned} W[y_1, y_2](t) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} t^{-1} & t^{\frac{1}{2}} \\ -t^{-2} & \frac{1}{2} t^{-\frac{1}{2}} \end{vmatrix} \\ &= \frac{1}{2} t^{-\frac{3}{2}} + t^{-\frac{3}{2}} \\ &= \frac{3}{2} t^{-\frac{3}{2}}, \end{aligned}$$

so we see that $W[y_1, y_2](t) \neq 0$ for $t > 0$. Thus, $y_1 = t^{-1}$ and $y_2 = t^{\frac{1}{2}}$ do indeed form a fundamental set of solutions.