MAT 22B Practice Final

August 30, 2022

Name: ____

Student ID: ____

- Carefully read each problem and make your arguments clear.
- You may not use any notes, books, or other outside materials.
- No electronic devices may be used to complete this exam.
- Work on your own exam. Any suspicions of violating the UC Davis Code of Academic Conduct will be reported to the Office of Student Support and Judicial Affairs.

Run LaTeX again to produce the table

1. (10 points) Consider the following differential equation

$$u'' + 3u' + 2u = 1 - e^{-3t}.$$

What happens to solutions as $t \to \infty$?

2. (10 points) Compute e^{At} where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Express your answer as $K = M + iP$ where M and P are real matrices.

3. (10 points) Solve the following initial value problem

$$y'' - 3y' + 2y = (1 + e^{-t})^{-1}, \quad y(0) = 0, \quad y'(0) = 1.$$

Hint: $\int \frac{e^{-2t}}{1+e^{-t}}\, dt = \ln|1 + e^{-t}| - e^{-t}$.

4. (10 points) Use the Laplace transform to solve the following initial value problem

$$y'' + 2y' + 2y = 2, \quad y(0) = 0, \quad y'(0) = 1.$$

5. Consider the following system of differential equations

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(a) (5 points) Write down the fundamental matrix, and show that the solutions you found in (a) form a fundamental set of solutions.

(b) (5 points) Find the general solution.

(c) (5 points) Suppose $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Determine the solution to the initial value problem.

6. (15 points) Find the general solution of the following system of differential equations

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \mathbf{x}.$$

7. (15 points) Find the general solution of the following system of differential equations

$$\mathbf{x}' = \begin{pmatrix} 3 & 1 \\ 0 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{2t} \\ 1+t \end{pmatrix}.$$

8. (15 points) Prove the following statement. If the complex valued function $\varphi(t) = u(t) + iv(t)$ is a solution to the linear differential equation

$$y^{(n)} = F(t, y, y', y'', \dots, y^{(n-1)}),$$

then $u(t)$ is a solution and $v(t)$ is a solution.

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\dfrac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. t^n; $\quad n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
4. $t^p, \ p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
5. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 6
6. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \lvert a\rvert$	Sec. 6.1; Prob. 8
8. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \lvert a\rvert$	Sec. 6.1; Prob. 7
9. $e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}, \ n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 19
16. $\displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 28