1 Basic Models and Direction Fields

What is a differential equation? Simply put, it is an equation of derivatives. Why would one want to study such equations? As we will see, they are convenient ways of describing the physical processes that we observe. That is, we observe something happen, and we hope to model the process using mathematics.

Consider the following example. All across campus, we see Turkeys, squirrels, and ducks. During certain times of the year, we encounter them more often than other times of the year, so we may initially hypothesize that the populations we observe on campus are not constant throughout the year. Let $p(t)$ be the population of turkeys at time t. What are some ways that the population can change?

- Migration
- Breeding
- Predation
- Accidents

Those are a few ways that the population can change, and we can broadly categorize these as births and deaths. Here is a differential equation

$$\frac{dp}{dt} = 4p - 10.$$

1.1 Direction Fields

Consider the arbitrary differential equation (DE)

$$\frac{dy}{dt} = f(t, y).$$

When we evaluate f, this tells us the value of $\frac{dy}{dt}$ at a point (t, y). That is, $f(t, y)$ tells us the slope of the cuve $y(t)$ at the points (t, y) in the ty-plane.

What are some potential problems with this model?

- The model can produce negative populations
- The population goes to $\pm\infty$

Now, we go back to the drawing board and cook-up an improved model.

1.2 Building a Model

When building a model, the general steps to follow are

1. Identify independent and dependent variables
2. Choose convenient units
3. Describe what you are modeling and basic principles governing the problem
4. Translate what you are modeling and the basic principles to the independent and dependent variables you identified in step 1

2 Constructing and Solving Your First Differential Equation

Consider an object of mass m falling with air resistance whose velocity at time t is given by $v(t)$. Newton's second law states that $F = ma$, where F is the net force acting on the object, and $a = \frac{dv}{dt}$ is the object's acceleration. The forces acting on a falling object are the gravitational force and air resistance. The gravitational force is $F_g = mg$, where g is the gravitational acceleration. Since the object is falling, air resistance acts in the opposite direction of the gravitational force, and we will assume air resistance is proportional to velocity and acts in the opposite direction of velocity. Therefore, $f_{\text{air}} = -\gamma v$, where γ is some constant of proportionality. Thus, Newton's second law gives

$$m\frac{dv}{dt} = mg - \gamma v,$$

and we have arrived at a differential equation describing the velocity of a falling object subject to air resistance. Now, let us solve our first differential equation. We have

$$\begin{aligned} m\frac{dv}{dt} &= mg - \gamma v \\ \frac{dv}{dt} &= g - \frac{\gamma}{m}v \\ \frac{dv}{dt} &= -\frac{\gamma}{m}\left(v - \frac{gm}{\gamma}\right) \\ \frac{v'}{v - \frac{gm}{\gamma}} &= -\frac{\gamma}{m} \\ \frac{d}{dt}\left(\ln\left|v - \frac{gm}{\gamma}\right|\right) &= -\frac{\gamma}{m} \\ \int \frac{d}{dt}\left(\ln\left|v - \frac{gm}{\gamma}\right|\right)\, dt &= \int -\frac{\gamma}{m}\, dt \\ \ln\left|v - \frac{gm}{\gamma}\right| &= -\frac{\gamma}{m}t + C_1 \end{aligned}$$

where C_1 is some arbitrary constant resulting from the integration that we performed. Since we want to know the velocity of the falling object, we find, after some manipulation, that

$$v(t) = \frac{gm}{\gamma} - C_2 e^{-\frac{\gamma}{m}t}.$$

The above expression is called the general solution, and its geometric representation is a family of curves called integral curves.

We may be unsatisfied that we are left with the consant C_2 in our expression, but we may determine C_2 using an initial condition. Suppose the object is released at time $t = 0$. Then, our initial condition is $v(0) = 0$. A differential equation with an initial value is called an initial value problem. For our problem at hand, the initial value problem is

$$m\frac{dv}{dt} = mg - \gamma v, \quad v(0) = 0.$$

Using the initial condition, we find that

$$v(0) = 0 = \frac{gm}{\gamma} - C_2$$

so $C_2 = \frac{gm}{\gamma}$. Thus, the solution to the initial value problem is

$$v(t) = \frac{gm}{\gamma}\left(1 - e^{-\frac{\gamma}{m}t}\right).$$

Notice that as $t \to \infty$, $v \to \frac{gm}{\gamma}$. That is, the object approaches some terminal velocity.