Example 0.1. *Find the general solution to*

$$\vec{x}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \vec{x}.$$

Example 0.2. *Consider the following system*

$$\vec{x}' = \begin{pmatrix} \alpha & 2 \\ -2 & -\lambda \end{pmatrix}.$$

What do solutions look like?

1 Fundamental Matrices

Suppose that $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ form a fundamenal set of solutions for

$$\vec{x}' = P(t)\vec{x}.$$

The matrix

$$\Psi = \begin{pmatrix} \vec{x}^{(1)} & \cdots & \vec{x}^{(n)} \end{pmatrix}$$

is called the fundamental matrix for the system. The general solution is

$$\vec{x}(t) = c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} + \cdots + c_n\vec{x}^{(n)},$$

or

$$\vec{x}(t) = \Psi(t)\vec{c}.$$

where $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$. The initial condition $\vec{x}(t_0) = \vec{x}^0$, so

$$\begin{aligned} \Psi(t_0)\vec{c} &= \vec{x}(t_0) \\ \Psi(t_0)\vec{c} &= \vec{x}^0 \\ \vec{c} &= \Psi^{-1}(t_0)\vec{x}^0. \end{aligned}$$

Therefore,

$$\vec{x}(t) = \Psi(t)\Psi^{-1}(t_0)\vec{x}^0.$$

1.1 The Matrix Exponential

Recall

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \cdots = 1 + \sum_{n=1}^{\infty} \frac{(at)^2}{n!}$$

and the scalar initial value problem

$$x' = ax, \quad x(0) = x_0$$

has solution

$$x = x_0 e^{at}.$$

Can we find something similar for

$$\vec{x}' = A\vec{x}, \quad \vec{x}(0) = \vec{x}^0?.$$

Let's use the matrix A in the Taylor series for the natural exponential. Then we have

$$e^{At} = I + \sum_{n=1}^{\infty} \frac{(At)^n}{n!}$$

so,

$$\begin{aligned}
\frac{d}{dt}e^{At} &= \frac{d}{dt}\left(I + \sum_{n=1}^{\infty}\frac{(At)^n}{n!}\right) \\
&= \sum_{n=1}^{\infty}\frac{n(At)^{n-1}}{n!}A \\
&= \sum_{n=1}^{\infty}\frac{(At)^{n-1}}{(n-1)!}A \\
&= A + \sum_{n=2}\frac{(At)^{n-1}}{(n-1)!}A \\
&= A + \sum_{n=1}^{\infty}A\frac{(At)^n}{n!} \\
&= A\left(I + \sum_{i=1}^{\infty}\frac{(At)^n}{n!}\right) \\
&= Ae^{At},
\end{aligned}$$

so

$$\vec{x} = e^{At}$$

satisfies the system of differential equations. Using the initial conditions, $\vec{x}(0) = \vec{x}^0$, we get

$$\vec{x}e^{At}\vec{x}^0.$$

1.2 Diagonalizable Matrices

Recall that a matrix A is said to be diagonalizable if we may write

$$A = TDT^{-1},$$

where D is a diagonal matrix whose entries are the eigenvalues of A,

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \\ 0 & \cdots & \cdots & \lambda_n & \end{pmatrix}$$

and

$$T = \begin{pmatrix} \vec{\xi}^{(1)} & \vec{\xi}^{(2)} & \cdots & \vec{\xi}^{(n)} \end{pmatrix},$$

where $\vec{\xi}^{(1)}, \ddots, \vec{\xi}^{(n)}$ are the corresponding eigenvectors. Now, the matrix exponential becomes

$$\begin{aligned}
e^{At} = I + \sum_{n=1}^{\infty}\frac{(At)^n}{n!} & \\
&= I + \sum_{n=1}^{\infty}\frac{(TDT^{-1})^n t^n}{n!} \\
&= I + \sum_{n=1}^{\infty}\frac{TD^nT^{-1}t^n}{n!} \\
&= T\left(I + \sum_{n=1}^{\infty}\frac{D^n t^n}{n!}\right)T^{-1} \\
&= Te^{Dt}T^{-1},
\end{aligned}$$

so

$$\vec{x} = Te^{Dt}T^{-1}\vec{x}^0.$$

Example 1.1. *Compute* e^{At} *where* $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$.