1 Classification of Differential Equations

1.1 PDE or ODE?

The first classification of differential equations that we will consider is whether we are given an ordinary differential equation or a partial differential equation. An ordinary differential equation (ODE) is an equation involving derivatives with repect to only one variable. For example,

$$\frac{d^2Q}{dt^2} + \frac{dQ}{dt} + Q = 0$$

or

$$Q'' + Q' + Q = 0$$

is an ODE. When we have an ODE with higher order derivatives, we use the notation

$$Q^{(n)} = \frac{d^nQ}{dt^n},$$

so

$$Q^{(1)} = \frac{dQ}{dt} = Q'.$$

When our unknown function has more than one independent variable, we are dealing with a partial differential equation. That is, we have an equation which involves derivatives with respect to more than one variable. A classical example of a partial differential equation is the heat equation

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}.$$

1.2 Systems of Differential Equations

Next, instead of being given a single differential equation, we may be given a system of differential equations. These arise when we have multiple dependent variables that interact. For example, the Lotka-Volterra (predator-prey) system is,

$$\frac{dx}{dt} = ax - \alpha xy$$
$$\frac{dy}{dt} = -cy + \gamma xy,$$

where x is the prey population and y is the predator population. The terms αxy and γxy describe the interaction between the predators and prey.

1.3 Order of a Differential Equation

The order of a differential equation is the order of the highest derivative appearing in the equation. For example,

$$y''' + yy' = t^4$$

is a third order ODE, because the highest derivative appearing is a third order derivative. A general n^{th} order ODE is of the form

$$F\left(t, y, y', \dots, y^{(n)}\right) = 0.$$

We will always assume that we may write the function F as

$$y^{(n)} = f\left(t, y, y', \dots, y^{(n-1)}\right).$$

1.4 Linear and Nonlinear Equations

The ODE

$$F\left(t, y, y', \ldots, y^{(n)}\right) = 0$$

is called linear if F is linear in y and its derivatives. A general n^{th} order linear ODE is of the form

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t).$$

Notice that the differential equation does not need to be linear in the independent variable. If the differential equation is not linear, then we refer to the differential equation as nonlinear. Nonlinear equations are much harder to solve, but sometimes we can approximate them using linear equations by linearing the differential equation. For example, if our differential equation contained a term involving $\sin\theta$, and θ was our dependent variable, we could use the approximation $\sin\theta \approx \theta$ to linearize the differential equation.

1.5 Solutions of Differential Equations

Loosely speaking, a solution of a differential equation, is a function that satisfies the equation. More precisely, consider the interval $\alpha < t < \beta$ and the n^{th} order ODE

$$y^{(n)} = f\left(t, y, y', \ldots, y^{(n-1)}\right).$$

The function $\varphi(t)$ is called a solution of the ODE if

$$\varphi^{(n)} = f\left(t, \varphi, \varphi', \ldots, \varphi^{(n-1)}\right)$$

for every $t \in (\alpha, \beta)$.

Consider the following differential equation

$$ty' - y = t^2.$$

We claim that $\varphi(t) = 3t + t^2$ is a solution to the differential equation. Differentiating our candidate function, we see that $\varphi'(t) = 3 + 2t$. Now, we substitute our candidate function into the differential equation, and we have

$$\begin{aligned} t(3 + 2t) - (3t + t^2) &= t^2 \\ t^2 &= t^2. \end{aligned}$$

We see that the two sides agree, so the equation is satisfied. Thus, we conclude that our claim is valid, and $\varphi(t) = 3t + t^2$ is indeed a solution to the differential equation.

Finding solutions to a given differential equation is often a monumental task, but with enough experience, one can start using one of the most useful techniques for solving a differential equation: making an ansatz. An ansatz is an educated guess which you can make, and if your ansatz satisfies the differential equation, then you've found a perfectly valid and acceptable solution!

Consider the following differential equation

$$y'' + y = 0.$$

Since we may rewrite the differential equation as

$$y'' = -y,$$

we see that we are looking for a function that produces the negative of itself after two differentiations. Two familiar functions that satisfy this property are sine and cosine.

1.6 Some remaining questions

We have seen that we can pull solutions out of thin air, but we may not always be able to do this. Additionally, we may find that there is more than one function that satisfies the differential equation. Is this acceptable? Finally, when can we explicitly determine the solution of a differential equation. The questions that we will eventually answer are:

1. Given a differential equation, when does a solution exist?
2. Given a differential equation, how many solutions does it have? Is the solution unique?
3. Given a differential equation, can we actually determine a solution?

2 First-Order Differential Equations

We will now focus our attention on differential equations of the form

$$\frac{dy}{dt} = f(t, y).$$

2.1 Linear Differential Equations and the Method of Integrating Factors

Our most general first-order linear ODE is

$$\frac{dy}{dt} + p(t)y = g(t).$$

Notice that we have written the differential equation so that it is in standard form. Here, p and g are given functions. Sometimes we write

$$P(t)\frac{dy}{dt} + Q(t)y = G(t).$$

By using the product rule, we can rewrite the differential equation in a form that will allow us to arrive at the solution upon direct integration.

Consider the following differential equation

$$\sin(t)\frac{dy}{dt} + \cos(t)\, y = t.$$

Upon recognizing the product rule on the left-hand side, we have

$$\begin{aligned}
\sin(t)\frac{dy}{dt} + \cos(t)\, y &= t \\
\frac{d}{dt}\left(\sin(t)\, y\right) &= t \\
\int \frac{d}{dt}\left(\sin(t)\, y\right)\, dt &= \int t\, dt \\
\sin(t)\, y &= \frac{t^2}{2} + C \\
y &= \frac{\frac{t^2}{2} + C}{\sin(t)}
\end{aligned}$$

We are not always so lucky and able to directly factor using the product rule. Multiplying the differential equation by an integrating factor $\mu(t)$ will allow us to rewrite the differential equation and integrate directly.

2.2 Derivation of the integrating factor

We have the first order linear ODE

$$\frac{dy}{dt} + p(t)y = g(t)$$

to which we will multiply our to be determined integrating factor $\mu(t)$.

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t).$$

We hope that the left-hand side may be written as $\frac{d}{dt}\left(r(t)y\right)$, and computing this derivative, we see that

$$\frac{d}{dt}\left(r(t)y\right) = r(t)\frac{dy}{dt} + r'(t)y.$$

Therefore, we want

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = r(t)\frac{dy}{dt} + r'(t)y,$$

and comparing the two sides, we see that requiring $r(t) = \mu(t)$ and $r'(t) = \mu(t)p(t)$ will give us the desired result. Now, we may determine $\mu(t)$ by solving the differential equation

$$\mu'(t) = \mu(t)p(t).$$

After staring at the above equation, we see it is of the same form as the very first differential equation we solved. Following the same steps, we have

$$\begin{aligned}
\mu'(t) &= \mu(t)p(t) \\
\frac{\mu'(t)}{\mu(t)} &= p(t) \\
\int \frac{d}{dt} \ln \mu(t)\, dt &= \int p(t)\, dt \\
\ln \mu(t) &= \int p(t)\, dt \\
\mu(t) &= e^{\int p(t)\, dt}
\end{aligned}$$

Now, we turn our attention back to the differential equation. After multiplying the differential equation by our integrating factor, we have

$$\begin{aligned}
e^{\int p(t)\, dt} \frac{dy}{dt} + e^{\int p(t)\, dt} p(t) y &= e^{\int p(t)\, dt} g(t) \\
\int \frac{d}{dt}\left(e^{\int p(t)\, dt} y\right)\, dt &= \int e^{\int p(t)\, dt} g(t)\, dt \\
e^{\int p(t)\, dt} y &= \int e^{\int p(t)\, dt} g(t)\, dt \\
y &= e^{-\int p(t)\, dt} \int e^{\int p(t)\, dt} g(t)\, dt,
\end{aligned}$$

and so our solution is

$$y = e^{-\int p(t)\, dt} \int e^{\int p(t)\, dt} g(t)\, dt.$$

Some remarks are in order:

1. There will be a constant of integration resulting from computing $\int e^{\int p(t)\, dt} g(t)\, dt$, so when we solve a specific differential equation, our solution will look like
$$y = \psi(t) e^{-\int p(t)\, dt} + C e^{-\int p(t)\, dt},$$
where $\psi(t) = \int e^{\int p(t)\, dt} g(t)\, dt$.
2. Notice that we did not concern ourselves with the constant of integration that should have appeared when we solved the differential equation for $\mu'(t)$. Do you see why we were allowed to ignore it?
3. The solution we arrived at may simply be memorized and applied, but I hope that the story told by our derivation of the integrating factor illustrates the motivation behind the technique.

Now, let us apply the method of integrating factors to the following initial value problem

$$(t^2 + 1)y' + y = t, \quad y(0) = 1.$$

Rewriting the differential equation in standard form, we have

$$y' + \frac{y}{t^2 + 1} = \frac{t}{t^2 + 1}.$$

Now, our integrating factor is

$$\begin{aligned}
\mu(t) &= e^{\int \frac{1}{t^2+1}\, dt} \\
&= e^{\tan^{-1}(t)}.
\end{aligned}$$

Multiplying the differential equation by $\mu(t)$, we have

$$\begin{aligned}
e^{\tan^{-1}(t)}y' + \frac{e^{\tan^{-1}(t)}}{t^2+1}y &= \frac{t}{t^2+1}e^{\tan^{-1}(t)} \\
\frac{d}{dt}\left(e^{\tan^{-1}(t)}y\right) &= \frac{t}{t^2+1}e^{\tan^{-1}(t)} \\
\int \frac{d}{dt}\left(e^{\tan^{-1}(t)}y\right)\, dt &= \int \frac{t}{t^2+1}e^{\tan^{-1}(t)}\, dt \\
e^{\tan^{-1}(t)}y &= \int_0^t \frac{s}{s^2+1}e^{\tan^{-1}(s)}\, ds + C \\
y &= e^{-\tan^{-1}(t)} \int_0^t \frac{s}{s^2+1}e^{\tan^{-1}(s)}\, ds + Ce^{-\tan^{-1}(t)}.
\end{aligned}$$

Now, to determine C, we use the initial condition which says that $y(0) = 1$, so

$$1 = y(0) = 0 + C.$$

Thus, the solution to the initial value problem is

$$y = e^{-\tan^{-1}(t)} \int_0^t \frac{s}{s^2+1}e^{\tan^{-1}(s)}\, ds + e^{-\tan^{-1}(t)}.$$

Remark: We see that we are left with an integral in our solution. This may seem unsatisfactory, but we may employ the use of a computer to approximate the definite integral which yields an approximation to the function y.