1 Separable Differential Equations

Consider the general first order ODE

$$\frac{dy}{dx} = f(x, y)$$

with initial condition $y(x_0) = y_0$. Suppose that we may write f as the product of two functions $g(x)$ and $h(y)$. That is, suppose we can separate f into two functions such that one of the functions only depends on x, and the other function only depends on y. Then, we have that

$$\begin{aligned} \frac{dy}{dx} &= f(x, y) \\ \frac{dy}{dx} &= g(x)h(y) \\ \frac{1}{h(y)}\frac{dy}{dx} &= g(x) \\ -g(x) + \frac{1}{h(y)}\frac{dy}{dx} &= 0. \end{aligned}$$

Now, let $G'(x) = -g(x)$ and $H'(y) = \frac{1}{h(y)}$. Then, we have

$$G'(x) + H'(y)\frac{dy}{dx} = 0,$$

and from the chain rule, we see that

$$\frac{d}{dx}\left(G(x) + H(y)\right) = G'(x) + H'(y)\frac{dy}{dx}.$$

Therefore,

$$\begin{aligned} G'(x) + H'(y)\frac{dy}{dx} &= 0 \\ \frac{d}{dx}\left(G(x) + H(y)\right) &= 0 \\ \int \frac{d}{dx}\left(G(x) + H(y)\right)\, dx &= \int 0\, dx \\ G(x) + H(y) - C &= 0, \end{aligned}$$

and so our solution is

$$G(x) + H(y) = C.$$

Now, if we are given an initial condition $y(x_0) = y_0$, we see that

$$G(x_0) + H(y_0) = C,$$

so

$$\begin{aligned} G(x) + H(y) &= G(x_0) + H(y_0) \\ G(x) - G(x_0) + H(y) - H(y_0) &= 0 \\ \int_{x_0}^{x} G'(s)\, ds + \int_{y_0}^{y} H'(s)\, ds &= 0. \end{aligned}$$

Now, recalling that $G'(x) = -g(x)$ and $H'(y) = \frac{1}{h(y)}$, we arrive at our solution

$$\int_{x_0}^{x} -g(s)\, ds + \int_{y_0}^{y} \frac{1}{h(s)}\, ds = 0,$$

which is an implicit representation of the solution. The final steps are to the perform the integrals and solve for y in terms of x.

Note: Often, determining an explicit formula is impossible, but you can use numerical methods to find approximate values of y for given values of x.

Now, let us solve the initial value problem

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}, \quad y(0) = 1.$$

We see that the differential equation is separable since

$$\frac{4x - x^3}{4 + y^3} = f(x)h(y),$$

where

$$f(x) = 4x - x^3, \quad h(y) = \frac{1}{4 + y^3}.$$

Now,

$$\begin{aligned}
\left(4 + y^3\right) \frac{dy}{dx} &= 4x - x^3 \\
\frac{d}{dx} \int_1^y 4 + s^3 \, ds &= 4x - x^3 \\
\int \frac{d}{dx} \int_1^y 4 + s^3 \, ds \, dx &= \int 4x - x^3 \, dx \\
\int_1^y 4 + s^3 \, ds &= \frac{4x^2}{2} - \frac{x^4}{4} + C \\
4s + \frac{s^4}{4}\bigg|_1^y &= 2x^2 - \frac{x^4}{4} + C \\
4y + \frac{y^4}{4} - \left(4 + \frac{1}{4}\right) &= 2x^2 - \frac{x^4}{4} + C \\
16y + y^4 - 17 &= 8x^2 - x^4 + C.
\end{aligned}$$

Now, our initial condition informs us that $y(0) = 1$, so

$$\begin{aligned}
16 + 1 - 17 &= C \\
C &= 0.
\end{aligned}$$

Thus, our solution to the initial value problem is

$$16y + y^4 = 8x^2 - x^4 + 17.$$

We are left with an implicit representation of our solution which makes sense for all values of x, but for what values of x is this solution valid? Examining the differential equation, we see that the derivative blows up $\left(\left|\frac{dy}{dx}\right| \to \infty\right)$ when $4 + y^4 = 0$ or $y = (-4)^{\frac{1}{3}}$. The corresponding values of x are solutions to

$$16(-4)^{\frac{1}{3}} + (-4)^{\frac{4}{3}} = 8x^2 - x^4 + 17,$$

and so $x \approx \pm 3.35$. Therefore, the domain of validity of the differential equation is $x \in [-3.35, 3.35]$.

2 Modeling with First-Order Differential Equations

Our motivation behind studying differential equations comes from their ability to model the physical phonomena that interest us. Our specializations vary, but the basic principles behind the construction of mathematical models follows the same ideas for each of us:

1. Construct the model
2. Analyze the model
3. Compare the model with our experiment or observation of interest

Constructing the model requies us to hypothesize about how our observations can be explained. Once we have built a model, we hope to analyze this model using the techniques we will study in this course. We can attempt to solve our governing equations (analytically or numerically), or we may attempt to gain qualitative insight about the predicted behavior. Once we understand how our model works, we then examine whether our model and observations align. If they do, then great! If they do not, we review the assumptions we made and see how we can improve our model to coincide with our experiment.

We will now construct a model using a first-order differential equation, and we will leave parameters in our differential equation, so that we may tune the model to our observations or test other hypothetical situations.

Lake Shasta in northern California is the state of California's largest reservoir, and it is formed by the Shasta Dam on the Sacramento river. We will create a model for the volume of water in Lake Shasta. Let $V(t)$ be the volume of water in the lake at time t, and we suppose that the volume of water in the lake is not constant. Let $r_1(t)$ be the inflow and $r_2(t)$ the outflow. Then,

$$\frac{dV}{dt} = r_1(t) - r_2(t).$$

How can we model $r_1(t)$?

- rain fall
- snow melt

We have not yet established a time scale, so let us choose the time scale to be years. Then,

$$r_1(t) = \cos(2\pi t)$$

gives us the desired shape for the inflow. Note that $t = 1$ is March of year 1, $t = 2$ is March of year 2, and so on. Next, we do not want negative flow, so let us add 1 to $r_1(t)$,

$$r_1(t) = \cos(2\pi t) + 1.$$

How should we model the outflow? For now, let us set $r_2(t) = c_1$. That is, there is a constant flow out of the lake. What are some potential problems we need to keep in mind when we choose c_1?

- If c_1 is too large, the lake could empty.
- If c_1 is too small, the lake could overflow.

What about initial conditions? The lake began storing water in 1944, so if we let $t = 0$ correspond to March 1944, then our initial condition is $V(0) = 0$. Thus, we have the initial value problem

$$\frac{dV}{dt} = \cos(2\pi t) + 1 - c_1, \quad V(0) = 0.$$

Now, we solve the initial value problem. We have

$$\int \frac{dV}{dt}\, dt = \int \cos(2\pi t) + 1 - c_1 \, dt$$
$$V(t) = \frac{1}{2\pi}\sin(2\pi t) + t - c_1 t + c_2.$$

Now, from the initial condition, we have

$$0 = V(0) = c_2.$$

Therefore, the solution to the initial value problem is

$$V(t) = \frac{1}{2\pi}\sin(2\pi t) + t - c_1 t.$$

We can find suitable values for c_1 by considering $V_{\max}$ and $V_{\min}$. That is, it may be undesirable to have the lake overflow or completely empty. You should check whether we can find c_1 so that the lake does not flow over the top of the dam.