1 Linear vs Nonlinear Differential Equations

Theorem. *Existence and uniqueness for first order linear differential equations. Let p and g be continuous functions on the interval $I : \alpha < t < \beta$ and $t_0 \in I$. Then, there is a unique function $y = \varphi(t)$ that satisfies the differential equation*

$$y' + p(t)y = g(t)$$

for each $t \in I$ and satsifies the initial condition $y(t_0) = y_0$.

Proof. We found that

$$\mu(t) = e^{\int p(t)\ dt}.$$

Since p is continuous for $\alpha < t < \beta$, μ is defined, differentiable, and nonzero. Next, μ and g are continuous, so μg is integrable and $\int \mu g\ dt$ is differentiable. Therefore,

$$y = \frac{1}{\mu(t)}\left(\int \mu(t)g(t)\ dt + C\right)$$

exists and is differentiable for $\alpha < t < \beta$. □

Theorem. *Existence and uniqueness for first order nonlinear differential equations. Let f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing (t_0, y_0). Then in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution to the initial value problem*

$$y' = f(t, y), \quad y(t_0) = y_0.$$

Note: The conditions in the theorem are sufficient, but not necessary. That is, we can make a weaker assumption on f. We can guarantee existence (but not uniqueness) by only assuming f is continuous.

Consider the initial value problem

$$y' = y^{\frac{1}{3}}, \quad y(0) = 0$$

for $t \geq 0$. We have that $f(t, y) = y^{\frac{1}{3}}$, so $\frac{\partial f}{\partial y} = \frac{1}{3}y^{-\frac{2}{3}}$. Notice that f is continuous everywhere, but $\frac{\partial f}{\partial y}$ does not exist when $y = 0$. Therefore, the previous theorem does not apply. Since f is continuous, a solution exists, but it is not necessarily unique.

Let us solve the initial value problem, and see the implication of not having a unique solution. Solving the differential equation, we have

$$\begin{aligned} y^{-\frac{1}{3}}y' &= 1 \\ \frac{3}{2}y^{\frac{2}{3}} &= t + c \\ y &= \left(\frac{2}{3}(t + c)\right)^{\frac{3}{2}}. \end{aligned}$$

Now, using the initial condition gives

$$0 = y(0) = \left(\frac{2}{3}c\right)^{\frac{3}{2}},$$

so $c = 0$. Therefore,

$$\varphi_1(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}}, \quad t \geq 0,$$

is a solution to the initial value problem. But also,

$$\varphi_2(t) = -\left(\frac{2}{3}t\right)^{\frac{3}{2}}, \quad t \geq 0$$

is a solution to the initial value problem. Notice that for any value of t_0,

$$y = \begin{cases} 0 & 0 \leq t < t_0 \\ \pm\left(\frac{2}{3}(t - t_0)\right)^{\frac{3}{2}}, & t \geq t_0 \end{cases}$$

is a solution to the initial value problem.

1.1 Interval of Existence

- Linear equations: The solution exists in the interval about $t = t_0$ in which p and g are continuous.
- Nonlinear equations: $y = \varphi(t)$ exists as long as $(t, \varphi(t))$ remains in the hypothesis region. Note that $\varphi(t)$ is not known ahead of time.

1.2 General Solution

- Linear equations: We can obtain all possible solutions by specifying the constant of integration.
- Nonlinear equations: There may be solutions that cannot be obtained by giving values to the constant.

1.3 Implicit Solutions

- Linear equations: $y = \varphi(t)$ (explicit)
- Nonlinear equations: $F(t, y) = 0$ (implicit)

2 Autonomous Differential Equations and Population Dynamics

A first order ODE in which the independent variable does not appear explicitly is called autonomous, and it is of the form

$$\frac{dy}{dt} = f(y).$$

This equation is separable, and we will examine this equation in the context of population dynamics.

2.1 Exponential Growth

Suppose the rate of change of a population is proportional to the current population. Then,

$$\frac{dy}{dt} = ry,$$

where r is the growth rate, and y is the population. If we specify that $y(0) = y_0$, then our solution to the initial value problem is $y = y_0 e^{rt}$. Now, let us examine the behavior of solutions.

- $r > 0$
  - If $y > 0$, then $\frac{dy}{dt} > 0$.
  - If $y < 0$, then $\frac{dy}{dt} < 0$.
  - If $y = 0$, then $\frac{dy}{dt} = 0$.
- $r < 0$
  - If $y > 0$, then $\frac{dy}{dt} < 0$
  - If $y < 0$, then $\frac{dy}{dt} > 0$
  - If $y = 0$, then $\frac{dy}{dt} = 0$
- $r = 0$ then $y = y_0$ for all t

From our plots of the direction field, we see that the solution trajectories exhibit different behaviors. When $r > 0$, solution trajectories move away from the equilibrium solution, but when $r < 0$, solution trajectories tend towards the equilibrium solution.

2.2 Logistic Growth

Now suppose that the growth rate depends on the population. Then we have

$$\frac{dy}{dt} = h(y)y.$$

Consider $h(y) = r - ay$, where $a > 0$. Then,

$$\frac{dy}{dt} = (r - ay)y,$$

or

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

where $K = \frac{r}{a}$. The parameter r is called the intrinsic growth rate, and it represents the rate of growth when there are no external influences. This differential equation is called the logistic equation.

2.3 Qualitative Analysis

Identifying equilibrium solutions gives us insight into the behavior of solutions. Equilibrium solutions are solutions for which $\frac{dy}{dt} = 0$ for all t. We may find these equilibrium solutions by determining the values of y for which $f(y) = 0$. Also, $\frac{dy}{dt} = f(y) = 0$, so we are looking for the critical points of y.

Now, let us analyze the logistic equation. We see that $f(y) = r\left(1 - \frac{y}{K}\right)y$, so we want to find the values of y such that

$$0 = r\left(1 - \frac{y}{K}\right)y.$$

We see that $y = K$ and $y = 0$ are two such solutions, so these are our two equilibrium solutions. Now, let us analyze the behavior of solutions when our initial condition is not an equilibrium solution. If $y \in (0, k)$, $\frac{y}{k} < 1$, so $1 - \frac{y}{k} > 0$. Therefore $\frac{dy}{dt} = f(y) > 0$. For $y \in (k, \infty)$, $\frac{y}{k} > 1$, so $1 - \frac{y}{k} < 0$. Therefore $\frac{dy}{dt} = f(y) < 0$. The y-axis is called the phase line, and note that previously, we plotted y vs t. We can determine the concavity of curves using the observation that

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\frac{dy}{dt} = \frac{d}{dt}f(y) = f'(y)\frac{dy}{dt} = f'(y)f(y).$$

Now, from our plot of the phase line, we see that solutions tend to K as $t \to \infty$, and K is called the saturation level or carrying capacity. The equilibrium $y = K$ is called a stable equilibrium, and $y = 0$ is called an unstable equilibrium.

2.4 Critical Threshold

Consider

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)y.$$

T is called the threshold level. If $y > T$, then we have growth. If $y < T$, then we have decay.

2.5 Logistic Growth with a Threshold

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{k}\right)y, \quad T < k.$$

In this model, we now have both a threshold and a saturation level.