1 Numerical Approxmations: Euler's Method

We have seen a few examples of first order differential equations that we can solve symbolically, but more often than not, we cannot solve the differential equation "by hand." We know that the IVP

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

has a unique solution $y = \varphi(t)$ if f and $\frac{\partial f}{\partial y}$ are continuous, so how can we "see" the solution?

One method is to draw a direction field and sketch some solution curves. This gives us good qualitative results, but we may want more concrete quantitative results. When we draw solution curves, what are we doing? This is exactly the idea behind the tangent line method or Euler's method!

Consider the IVP

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

The solution passes through (t_0, y_0), and the slope at (t_0, y_0) is $f(t_0, y_0)$. Therefore, the tangent line to the solution at (t_0, y_0) is

$$y = y_0 + f(t_0, y_0)(t - t_0),$$

and as long as we stay close to $t = t_0$, this is a good approximation.

Say we want a tangent line approximation at $t = t_1$. Let's use the tangent line at (t_0, y_0) to approximate y_1, so

$$y_1 = y_0 + f(t_0, y_0)(t_1 - t_0).$$

Next, we want to find (t_2, y_2), so using the tangent line at (t_1, y_1), we have

$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1),$$

and so

$$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n).$$

This procedure produces a sequence $y_0, y_1, \ldots$ at $t_0, t_1, \ldots$ which gives an approximation to the solution.

If we assume a uniform step size h between $t_0, t_1, t_2, \ldots$, then $t_{n+1} = t_n + h$, so

$$y_{n+1} = y_n + hf(t_n, y_n).$$

Euler's method is a sequence of tedious computations, so rather than doing this method by hand, we will ask a computer to do the computations for us.

The numerical method above is called (forward/explicit) Euler's method. There is also what's called backward/implicit Euler's method. Instead of evaluating $f(t, y)$ at the current value (t_n, y_n), we evaluate f at (t_{n+1}, y_{n+1}). Then, our method becomes

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}).$$

For a specific funciton f, we hope to rearrange the equation so that we may explicitly write y_{n+1} in terms of y_n, h, and t_{n+1}.

Set up the Euler iteration scheme for the IVP

$$\frac{dy}{dt} = t^2 + 5y, \quad y(0) = 1.$$

Explicit:

$$\begin{aligned} y_{n+1} &= y_n + hf(t_n, y_n) \\ &= y_n + h(t_n^2 + 5y_n) \\ &= y_n(1 + 5h) + ht_n^2. \end{aligned}$$

Implicit:

$$\begin{aligned} y_{n+1} &= y_n + hf(t_{n+1}, y_{n+1}) \\ y_{n+1} &= y_n + h(t_{n+1}^2 + 5y_{n+1}) \\ y_{n+1}(1-5h) &= y_n + ht_{n+1}^2 \\ y_{n+1} &= \frac{y_n + ht_{n+1}^2}{1-5h} \\ y_{n+1} &= \frac{y_n + h(t_n+h)^2}{1-5h}. \end{aligned}$$

2 The Existence and Uniqueness Theorem

Theorem. *If f and $\frac{\partial f}{\partial y}$ are continuous in a rectangle $R : |t| \le a$, $|y| \le b$, then there is some interval $|t| \le h \le a$ in which there exists a unique solution $y = \varphi(t)$ of the initial value problem*

$$y' = f(t, y), \quad y(0) = 0.$$

Note that if we have the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0,$$

we can make a change of variables so that the initial point (t_0, y_0) is the origin.

Now, we will examine elements of tools used to prove the existence and uniqueness theorem. Suppose $y = \varphi(t)$ satisfies the initial value problem. Then $f(t, y) = f(t, \varphi(t))$, and we see that we have written f so that is only depends on t. Now,

$$\begin{aligned} y' &= f(t, y) \\ y' &= f(t, \varphi(t)) \\ \int_0^t y' \, dt &= \int_0^t f(s, \varphi(s)) \, ds \\ y(t) - y(0) &= \int_0^t f(s, \varphi(s)) \, ds \\ y(t) &= \int_0^t f(s, \varphi(s)) \, ds. \end{aligned}$$

The expression for $y(t)$ at which we have arrived involves the unknown function φ, and it is called an integral equation. The important point is that the integral equation and initial value problem are equivalent.

2.1 Picard's Iteration Method (Method of Successive Approximations)

We will now examine Picard iterates which are a method for showing that the integral equation has a unique solution. We will generate a sequence of functions $\{\varphi_n(t)\}$ as follows:

1. Choose φ_0. The simplest choice is $\varphi_0(t) = 0$.
2. We get φ_1 by using φ_0 in the integral equation

$$\varphi_1(t) = \int_0^t f(s, \varphi_0(s)) \, ds.$$

3. We get φ_2 by using φ_1

$$\varphi_2(t) = \int_0^t f(s, \varphi_1(s)) \, ds,$$

and so on. We get φ_n by the expression

$$\varphi_n(t) = \int_0^t f(s, \varphi_{n-1}(s)) \, ds.$$

We now have a sequence of functions, and each function in the sequence satisfies the initial condition, but they may not satisfy the differential equation. If at some point, say $n = k$, we have $\varphi_{k+1}(t) = \varphi_k(t)$, then φ_k is a solution of the integral equation.

Now, to prove the existence and eniqueness theorem, we need to know:

1. Do all members of $\{\varphi_n\}$ exist?
2. Does $\{\varphi_n\}$ converge?
3. What are the properties of the limit function?
4. Is this the only solution?

2.2 1. Do all members of $\{\varphi_n\}$ exist?

f and $\frac{\partial f}{\partial y}$ are continuous in the rectangle $R : |t| \leq a, |y| \leq b$, so the danger is that $y = \varphi_k(t)$ may be outside of R. We need to restrict t to an interval smaller than $|t| \leq a$. Since f is continuous on a closed, bounded region, there exists a constant $M > 0$ such that

$$|f(t, y)| \leq M$$

for $(t, y) \in R$. Now,

$$\varphi'_{k+1} = f(t, \varphi_k) \leq M,$$

so the point $(t, \varphi_{k+1}(t)) \in R$. Therefore $(t, \varphi_{k+1}(t)) \in R$ as long as $|t| \leq \frac{b}{M}$. Choose $h = \min\left\{a, \frac{b}{M}\right\}$.

2.3 Does $\{\varphi_n\}$ converge?

We see that

$$\varphi_n(t) = \varphi_1 + (\varphi_2 - \varphi_1) + \cdots + (\varphi_n - \varphi_{n-1})$$

is the partial sum of the series

$$\varphi_1 + \sum_{k=1}^{\infty} (\varphi_{k+1} - \varphi_k).$$

Therefore, if the series converges, then the sequence $\{\varphi_n\}$ converges and we let

$$\varphi(t) = \lim_{n\to\infty} \varphi_n(t).$$

2.4 What are the properties of the limit function?

We want to know if φ is continuous. If each φ_n is continuous and $\{\varphi_n\}$ "converges uniformly," then φ is continuous. We have

$$\varphi_{n+1}(t) = \int_0^t f(s, \varphi_n(s))\, ds$$

and so

$$\begin{aligned}
\varphi(t) &= \lim_{n\to\infty} \int_0^t f(s, \varphi_n(s))\, ds \\
&= \int_0^t \lim_{n\to\infty} f(s, \varphi_n(s))\, ds \\
&= \int_0^t f(s, \lim_{n\to\infty} \varphi_n(s))\, ds \\
&= \int_0^t f(s, \varphi(s))\, ds.
\end{aligned}$$

2.5 Is this the only solution?

Suppose $y_1 = \varphi(t)$ and $y_2 = \psi(t)$ are both solutions. Then

$$\begin{aligned} |y_1 - y_2| &= |\varphi(t) - \psi(t)| \\ &= \left| \int_0^t f(s, \varphi(s))\, ds - \int_0^t f(s, \psi(s))\, ds \right| \\ &= \left| \int_0^t f(s, \varphi(s)) - f(s, \psi(s))\, ds \right| \\ &\leq \int_0^t |f(s, \varphi(s)) - f(s, \psi(s))|\, ds \\ &\leq \int_0^t L\, |\varphi(s) - \psi(s)|\, ds \end{aligned}$$

for some constant L (Lipschitz consant). Let

$$A(t) = \int_0^t |\varphi(s) - \psi(s)|\, ds.$$

Then, $A(0) = 0$, $A(t) \geq 0$ for $t \geq 0$, and

$$A'(t) = |\varphi(t) - \psi(t)|\, .$$

Now,

$$|\varphi(t) - \psi(t)| \leq L \int_0^t |\varphi(s) - \psi(s)|\, ds,$$

so

$$\begin{aligned} A'(t) &\leq LA(t) \\ A'(t) - LA(t) &\leq 0. \end{aligned}$$

Now, multiplying by e^{-Lt}, we get

$$\frac{d}{dt}\left(e^{-Lt}A(t)\right) \leq 0$$

and so

$$e^{-Lt}A(t) \leq 0.$$

Hence,

$$A(t) \leq 0.$$

We now have that $A(0) = 0$, $A(t) \geq 0$ for $t \geq 0$ and $A(t) \leq 0$ for $t \geq 0$, so we must have that $A(t) = 0$ for $t \geq 0$. Hence, $A'(t) = 0$, so

$$|\varphi(t) - \psi(t)| = 0.$$

Ergo, $\varphi(t) = \psi(t)$ for $t \geq 0$, and so the solution $\varphi(t)$ is unique.