1 First-Order Difference Equations

So far, we have considered continuous models, but now we explore discrete models which will lead us to difference equations. A first-order difference equation is of the form

$$y_{n+1} = f(n, y_n), \quad n = 0, 1, 2, \dots .$$

The difference equation is called first-order, because f is a function of y_n and not any other previous values such as $y_{n-1}, y_{n-2}, \dots$. The difference equation is linear if f is a linear function of y_n; otherwise it is nonlinear. A solution is a sequence of numbers $y_0, y_1, \dots$ that satisfies the equation for each n. The initial condition $y_0 = \alpha$ specifies the first term of the solution sequence, and we may compute the terms that follow.

1.1 Equilibrium Solutions

Suppose

$$y_{n+1} = f(y_n), \quad n = 0, 1, 2, \dots ,$$

where f only depends on y_n. Then,

$$\begin{aligned} y_1 &= f(y_0) \\ y_2 &= f(y_1) = f(f(y_0)) \\ y_3 &= f(y_2) = f(f(f(y_0))) = f^3(y_0), \end{aligned}$$

and so

$$y_n = f^n(y_0).$$

We may want to understand the behavior of y_n as $n \to \infty$, and equilibrium solutions will give us a picture of how solutions behave. If y_n has the same value for all n, y_n is called an equilibrium solution, and such a solution satisfies the equation

$$y_n = f(y_n).$$

1.2 Linear Equations

Consider the difference equation

$$y_{n+1} = \rho y_n + b_n, \quad n = 0, 1, 2, \dots .$$

First, let's solve the equation in terms of the initial value y_0. To do this, let's compute $y_1, y_2, \dots$, and see if we spot a pattern developing. We have

$$y_1 = \rho y_0 + b_0,$$

$$\begin{aligned} y_2 &= \rho y_1 + b_1 \\ &= \rho\left(\rho y_0 + b_0\right) + b_1 \\ &= \rho^2 y_0 + \rho b_0 + b_1, \end{aligned}$$

and

$$\begin{aligned} y_3 &= \rho y_2 + b_2 \\ &= \rho\left(\rho^2 y_0 + \rho b_0 + b_1\right) + b_2 \\ &= \rho^3 y_0 + \rho^2 b_0 + \rho b_1 + b_2. \end{aligned}$$

After comparing y_1, y_2, and y_3, we see that

$$\begin{aligned} y_n &= \rho^n y_0 + \rho^{n-1} b_0 + \cdots + \rho b_{n-2} + b_{n-1} \\ &= \rho^n y_0 + \sum_{j=0}^{n-1} \rho^{n-1-j} b_j. \end{aligned}$$

Now, let us suppose that $b_n = b \neq 0$ for all n. Then,

$$\begin{aligned} y_n &= \rho^n y_0 + b \sum_{j=0}^{n-1} \rho^{n-1-j} \\ &= \rho^n y_0 + b \frac{1-\rho^n}{1-\rho} \\ &= \rho^n \left(y_0 - \frac{b}{1-\rho} \right) + \frac{b}{1-\rho}. \end{aligned}$$

Now, let's determine the long-time behavior of y_n.

1. If $|\rho| < 1$, then $y_n \to \frac{b}{1-\rho}$ as $n \to \infty$.
2. Suppose $|\rho| > 1$.
   (a) If $y_0 = \frac{b}{1-\rho}$, then $y_n = \frac{b}{1-\rho}$ for all n.
   (b) If $y_0 \neq \frac{b}{1-\rho}$, then there is no (finite) limit.
3. If $\rho = -1$, there is no limit.
4. If $\rho = 1$, we need to go back to the difference equation,

$$\begin{aligned} y_{n+1} &= \rho y_n + b \\ &= y_n + b \end{aligned}$$

   and so $y_n \to \infty$ as $n \to \infty$.

1.3 Nonlinear Equations

Consider the logistic difference equation

$$y_{n+1} = \rho y_n \left(1 - \frac{y_n}{k} \right)$$

or

$$u_{n+1} = \rho u_n \left(1 - u_n \right),$$

where we used the change of variable

$$u_n = \frac{y_n}{k}.$$

Now let's find equilibrium solutions,

$$\begin{aligned} u_n &= \rho u_n \left(1 - u_n \right) \\ \rho u_n^2 + u_n(1-\rho) &= 0 \\ u_n \left(\rho u_n + 1 - \rho \right) &= 0, \end{aligned}$$

and so $u_n = 0$ and $u_n = \frac{\rho - 1}{\rho}$ are our equilibrium solutions. Now, let us draw the associated cobweb (stairstep) diagram.

2 Second-Order Linear Differential Equations

After exploring first-order differential equations, we now turn our focus to second-order linear differential equations. Consider the following differential equation,

$$\frac{d^2 y}{dt^2} = f\left(t, y, \frac{dy}{dt} \right).$$

This is a second-order differential equation, and it is linear if

$$f\left(t, y, \frac{dy}{dt} \right) = g(t) - p(t)\frac{dy}{dt} - q(t) y,$$

where g, p, and q are given functions of t. Our general second-order linear differential equation is

$$y'' + p(t)y' + q(t)y = g(t).$$

The initial value problem now requires us to specify

$$y(t_0) = y_0$$

and

$$y'(t_0) = y_0'.$$

That is, we now require specifing an initial point (t_0, y_0) and an initial slope y_0'. The second-order differential equation is called homogeneous if $g(t) = 0$ for all t. Otherwise, it is called nonhomogeneous. Solving the nonhomogeneous problem requires us to first solve the homogeneous problem, so we now examine the homogeneous case.