1 Homogeneous Differential Equations with Constant Coefficients

Consider

$$ay'' + by' + cy = 0,$$

where a, b, and c are given constants. Recall that we have already seen how to solve the differential equation! Solving the second-order linear constant coefficient homogeneous differential equation relies on two key ideas: a good ansatz and linearity.

1.1 Idea I: Ansatz and the Characteristic Equation

Suppose $y = e^{rt}$ is a solution to the differential equation. Then

$$\begin{aligned} ay'' + by' + cy &= a\left(e^{rt}\right)'' + b\left(e^{rt}\right)' + ce^{rt} \\ &= ar^2 e^{rt} + bre^{rt} + ce^{rt} \\ &= e^{rt}\left(ar^2 + br + c\right), \end{aligned}$$

and so we want

$$e^{rt}\left(ar^2 + br + c\right) = 0.$$

Now, $e^{rt} \neq 0$, so we require

$$ar^2 + br + c = 0.$$

This equation is called the characteristic equation of the differential equation and has two roots, r_1 and r_2. Depending on the constants a, b, and c, we may encounter roots which are real and distinct, real and repeated, or complex conjugates. For now, let us assume r_1 and r_2 are distinct real roots. Then,

$$y_1 = e^{r_1 t}$$

and

$$y_2 = e^{r_2 t}$$

are both solutions to the differential equation.

1.2 Idea II: Linearity

If y_1 and y_2 are solutions of the differential equation, then $y = c_1 y_1 + c_2 y_2$ solves the differential equation. Why? We see that

$$\begin{aligned} ay'' + by' + cy &= a\left(c_1 y_1 + c_2 y_2\right)'' + b\left(c_1 y_1 + c_2 y_2\right)' + c\left(c_1 y_1 + c_2 y_2\right) \\ &= ac_1 y_1'' + bc_1 y_1' + cc_1 y_1 + ac_2 y_2'' + bc_2 y_2' + cc_2 y_2 \\ &= c_1\left(ar_1^2 + br_1 + c\right)e^{r_1 t} + c_2\left(ar_2^2 + br_2 + c\right)e^{r_2 t} \\ &= 0, \end{aligned}$$

since r_1 and r_2 are roots of $p(r) = ar^2 + br + c$.

1.3 The Initial Value Problem

We have confirmed that

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

solves the differential equation, and we call this the general solution. We obtain a specific solution to the initial value problem by determining the constants c_1 and c_2 using the initial values

$$y(t_0) = y_0$$

and

$$y'(t_0) = y_0'.$$

Consider the initial value problem

$$4y'' - 8y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{2}.$$

First, the characteristic equation is

$$p(r) = 4r^2 - 8r + 3.$$

Next, we find the roots of $p(r)$,

$$\begin{aligned} 4r^2 - 8r + 3 &= 0 \\ (2r-1)(2r-3) &= 0, \end{aligned}$$

and so $r_1 = \frac{3}{2}$ and $r_2 = \frac{1}{2}$. Thus, our general solution is

$$y = c_1 e^{\frac{3}{2}t} + c_2 e^{\frac{1}{2}t}.$$

Now, we determine c_1 and c_2 using the initial conditions. We have

$$\begin{aligned} 2 &= y(0) \\ &= c_1 + c_2, \end{aligned}$$

so

$$c_1 = 2 - c_2.$$

We now have that

$$y = (2 - c_2)e^{\frac{3}{2}t} + c_2 e^{\frac{1}{2}t},$$

so

$$y' = \frac{3}{2}(2 - c_2)e^{\frac{3}{2}t} + \frac{1}{2}c_2 e^{\frac{1}{2}t}.$$

Using the second initial condition, we have

$$\begin{aligned} \frac{1}{2} &= y'(0) \\ \frac{1}{2} &= \frac{3}{2}(2 - c_2) + \frac{1}{2}c_2 \\ 1 &= 3(2 - c_2) + c_2 \\ -5 &= -2c_2 \\ c_2 &= \frac{5}{2}, \end{aligned}$$

and so

$$\begin{aligned} c_1 &= 2 - c_2 \\ &= 2 - \frac{5}{2} \\ &= -\frac{1}{2}. \end{aligned}$$

Thus, our solution to the initial value problem is

$$y = -\frac{1}{2}e^{\frac{3}{2}t} + \frac{5}{2}e^{\frac{1}{2}t}.$$

1.4 Long-time Behavior

Our general solution to the second-order linear constant coefficient differential equation is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

What happens as $t \to \infty$? We see that the behavior of the solution as $t \to \infty$ depends on the roots of the characteristic equation. We will return to this question after figuring out how our general solution presents itself when we come across complex and real repeated roots.

2 Solutions of the Homogeneous Equations and the Wronskian

Let p and q be continuous functions on an open interval I. Consider the differential operator L defined by

$$L[\varphi] = \varphi'' + p\varphi' + q\varphi,$$

where φ is some function. Here, L operates on φ. That is to say, φ acts as the input to the operator L. The value of $L[\varphi]$ at a point t is

$$L[\varphi](t) = \varphi''(t) + p(t)\varphi'(t) + q(t)\varphi(t).$$

We will examine the second-order linear differential equation

$$L[\varphi](t) = 0,$$

with initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0.$$

We want to know whether the initial value problem has a solution, whether it has more than one solution and how solutions are structured and formed.

Theorem. *(Existence and Uniqueness) Let p, g, and q be continuous in an open interval I containing t_0. Then the initial value problem*

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0',$$

has a unique solution $y = \varphi(t)$ in the interval I.

Theorem. *If y_1 and y_2 are solutions to the differential equation*

$$L[y] = y'' + p(t)y' + q(t) = 0,$$

then the linear combination $c_1 y_1 + c_2 y_2$ is also a solution for any c_1 and c_2.

The proof of the previous theorem follows from the linearity of L. Now, when can c_1 and c_2 be chosen to satisfy the initial conditions? We require

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= y_0', \end{aligned}$$

or in matrix form

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}.$$

When does this system have a unique solution? Let

$$W(t_0) = \det\left(\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix} \right).$$

If $W(t_0) \neq 0$, then we are guaranteed a unique solution $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, and W is called the Wronskian of the solutions y_1 and y_2.

Theorem. *Suppose y_1 and y_2 are two solutions of*

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

It is possible to choose c_1 and c_2 so that

$$y = c_1y_1(t) + c_2y_1(t)$$

satisfies the differential equation and initial conditions if and only if the Wronskian

$$W[y_1, y_2] = y_1y_2' - y_1'y_2$$

is not zero at t_0.

Consider the differential equation

$$y'' + 5y' + 6y = 0.$$

Find the Wronskian of y_1 and y_2. The characteristic equation is

$$\begin{aligned} p(r) &= r^2 + 5r + 6 \\ &= (r + 3)(r + 2), \end{aligned}$$

so our roots are $r_1 = -3$ and $r_2 = -2$. Therefore, our solutions are

$$y_1 = e^{-2t}$$

and

$$y_2 = e^{-3t}.$$

Now,

$$\begin{aligned} W[y_1, y_2](t) &= \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} \\ &= -3e^{-5t} + 2e^{-5t} \\ &= -e^{-5t}. \end{aligned}$$

We see that $W[y_1, y_2](t) \neq 0$ for all t, so any initial condition can be specified at any initial time t. The expression

$$y = c_1y_1(t) + c_2y_2(t)$$

is called the general solution of $L[y] = 0$, and y_1 and y_2 are said to form a fundamental set of solutions if their Wronskian is nonzero for all t. Notice that we have a nonzero Wronskian if and only if y_1 and y_2 are linearly independent.

Show that $y_1 = t^{\frac{1}{2}}$ and $y_2 = \frac{1}{t}$ form a fundamental set of solutions of

$$2t^2y'' + 3ty' - y = 0, \quad t > 0.$$

First, we need to show that y_1 and y_2 satisfy the differential equation. Using y_1 in the differential equation, we have

$$\begin{aligned} 2t^2y_1'' + 3ty_1' - y_1 &= 2t^2(t^{\frac{1}{2}})'' + 3t(t^{\frac{1}{2}})' - t^{\frac{1}{2}} \\ &= 2t^2\left(\frac{1}{2}t^{-\frac{1}{2}}\right)' + 3t\frac{1}{2}t^{-\frac{1}{2}} - t^{\frac{1}{2}} \\ &= 2t^2\left(-\frac{1}{4}\right)t^{-\frac{3}{2}} + \frac{3}{2}t^{\frac{1}{2}} - t^{\frac{1}{2}} \\ &= -\frac{1}{2}t^{\frac{1}{2}} + \frac{1}{2}t^{\frac{1}{2}} \\ &= 0, \end{aligned}$$

so y_1 is a solution. Next, we use y_2 in the differential equation,

$$\begin{aligned}
2t^2y_2'' + 3ty_2' - y_2 &= 2t^2\left(\frac{1}{t}\right)'' + 3t\left(\frac{1}{t}\right)' - \frac{1}{t} \\
&= 2t^2\left(-\frac{1}{t^2}\right)' + 3t\left(-\frac{1}{t^2}\right) - \frac{1}{t} \\
&= 2t^2\frac{2}{t^3} - \frac{3}{t} - \frac{1}{t} \\
&= \frac{4}{t} - \frac{4}{t} \\
&= 0,
\end{aligned}$$

so y_2 is a solution. Now, we compute the Wronskian

$$\begin{aligned}
W[y_1, y_2](t) &= \begin{vmatrix} t^{\frac{1}{2}} & \frac{1}{t} \\ \frac{1}{2}t^{-\frac{1}{2}} & -\frac{1}{t^2} \end{vmatrix} \\
&= t^{\frac{1}{2}}\left(-t^{-2}\right) - \frac{1}{2}t^{-\frac{1}{2}}t^{-1} \\
&= -t^{-\frac{3}{2}} - \frac{1}{2}t^{-\frac{3}{2}} \\
&= -\frac{3}{2}t^{-\frac{3}{2}}.
\end{aligned}$$

We see that $W[y_1, y_2](t) \neq 0$ for $t > 0$, so y_1 and y_2 form a fundamental set of solutions. Thus, the general solution to the differential equation is

$$y(t) = c_1t^{\frac{1}{2}} + c_2\frac{1}{t}.$$

Theorem. *If $y = u(t) + iv(t)$ is a complex-valued solution of $L[\varphi] = 0$, then its real part u and imaginary part v are also solutions.*

The proof follows form the linearity of L, that $L[y] = 0$, and the fact that if $a + ib = 0$, then $a = 0$ and $b = 0$.

Theorem. *(Abel's Theorem) If y_1 and y_2 are solutions of*

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous on I, then the Wronskain is

$$W[y_1, y_2](t) = Ce^{-\int p(t)\,dt},$$

where C is a constant depending on y_1 and y_2, but not on t. W is either zero for all t in I or never zero in I.

3 Complex Roots of the Characteristic Equation

Our second-order constant coefficient homogeneous differential equation is

$$ay'' + by' + cy = 0,$$

where a, b, and c are constants. The characteristic equation is

$$p(r) = ar^2 + br + c,$$

so

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

What happens when $b^2 - 4ac < 0$? We have roots r_1 and r_2 which are complex conjugates of each other

$$r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu,$$

where λ and μ are real numbers. Our solutions are

$$y_1 = e^{(\lambda + i\mu)t}$$

and

$$y_2 = e^{(\lambda - i\mu)t}.$$

What does Euler's number raised to a complex power even mean?

3.1 Euler's Formula

Recall the Taylor (Maclaurin) series for e^t about $t = 0$,

$$\begin{aligned} e^t &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \\ &= 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \cdots . \end{aligned}$$

Now,

$$\begin{aligned} e^{it} &= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \\ &= 1 + it - \frac{t^2}{2!} - \frac{it^3}{3!} + \frac{t^4}{4!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \\ &= \cos(t) + i \sin(t). \end{aligned}$$

Thus,

$$e^{it} = \cos(t) + i \sin(t),$$

and this is called Euler's formula. Now,

$$\begin{aligned} y_1 &= e^{(\lambda + i\mu)t} \\ &= e^{\lambda t} e^{i\mu t} \\ &= e^{\lambda t} \left(\cos(\mu t) + i \sin(\mu t) \right) \end{aligned}$$

and

$$\begin{aligned} y_2 &= e^{(\lambda - i\mu)t} \\ &= e^{\lambda t} e^{-i\mu t} \\ &= e^{\lambda t} \left(\cos(\mu t) - i \sin(\mu t) \right), \end{aligned}$$

where we recalled that sine is an odd function

$$\sin(-t) = -\sin(t),$$

and cosine is an even function

$$\cos(-t) = \cos(t).$$