1 Nonhomogeneous Equations

Let

$$L[y] = y'' + p(t)y' + q(t)y,$$

be our second-order differential operator. We will now consider the nonhomogeneous second-order linear differential equation $L[y] = g(t)$ or

$$y'' + p(t)y' + q(t)y = g(t). \tag{1}$$

When $g(t) = 0$, we call the resulting differential equation the homogeneous differential equation corresponding to (1).

Theorem. *If Y_1 and Y_2 are solutions of the nonhomogeneous linear differential equation $L[y] = g(t)$, then $Y_1 - Y_2$ is a solution of the homogeneous differential equation $L[y] = 0$. If y_1 and y_2 form a fundamental set of solutions to $L[y] = 0$, then*

$$Y_1 - Y_2 = c_1y_1 + c_2y_2.$$

Proof. Since Y_1 and Y_2 satisfy $L[y] = g(t)$, we have

$$\begin{aligned} L[Y_1 - Y_2] &= L[Y_1] - L[Y_2] \\ &= g(t) - g(t) \\ &= 0, \end{aligned}$$

so $Y_1 - Y_2$ is a solution of $L[y] = 0$. Since any solution of $L[y] = 0$ is a linear combination of the fundamental set of solutions, we must have that

$$Y_1 - Y_2 = c_1y_1 + c_2y_2.$$

□

Theorem. *The general solution of $L[y] = g(t)$ is of the form*

$$y = c_1y_1 + c_2y_2 + Y,$$

where y_1 and y_2 form a fundamental set of solutions to $L[y] = 0$, and Y is any solution of $L[y] = g(t)$.

1.1 Solving $L[y] = g(t)$

1. Find the general solution to the homogeneous equation $L[y] = 0$ which is called the complementary solution,

$$y_c = c_1y_1 + c_2y_2.$$

2. Find the solution to $L[y] = g(t)$ which is called the particular solution y_p.
3. The general solution to the nonhomogeneous problem is

$$y = y_c + y_p.$$

We have seen how to find y_c, so how do we find y_p?

2 The Method of Undetermined Coefficients

To determine y_p, we will make an ansatz based on the form of $g(t)$. Our ansatz will involve to-be-determined constants which will be determined by the requirement that $L[y_p] = g(t)$.

Example 2.1. *Determine the particular solution to*

$$y'' - 3y' + 4y = 3e^{2t}.$$

We want to find y_p so that

$$y_p'' - 3y_p' + 4y_p = 3e^{2t},$$

so let us try $y_p = Ae^{2t}$. *Then,*

$$\begin{aligned} 4Ae^{2t} - 6Ae^{2t} + 4Ae^{2t} &= 3e^{2t} \\ 2Ae^{2t} &= 3e^{2t} \\ A &= \frac{3}{2}, \end{aligned}$$

so $y_p = \frac{3}{2}e^{2t}$.

The same idea holds when $g(t)$ is of a different form. Some common forms of $g(t)$ that we will consider are

- Exponential: Guess y_p is proportional to some exponential,
- Sine or Cosine: Guess y_p is a linear combination of sine and cosine,
- Degree n polynomial: Guess y_p is an n degree polynomial.

This same idea holds when you have a product or sum of the above three types of functions. If $g(t) = g_1(t) + g_2(t)$ and $L[Y_1] = g_1(t)$ and $L[Y_2] = g_2(t)$, then $Y_1 + Y_2$ is a solution to $L[y] = g(t)$. That is, determining the particular solution can be broken up into several smaller problems when $g(t)$ can be expressed as a sum of exponential, trigonometric, or polynomial functions.

Example 2.2. *Find a particular solution of*

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t\cos(2t).$$

We can split the problem up into

$$y'' - 3y' - 4y = 3e^{2t},$$

$$y'' - 3y' - 4y = 2\sin t,$$

and

$$y'' - 3y' - 4y = -8e^t\cos(2t).$$

First, we guess that $y_{p_1} = Ae^{2t}$. *Now,*

$$\begin{aligned} y_{p_1}'' - 3y_{p_1}' - 4y_{p_1} &= 3e^{2t} \\ 4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} &= 3e^{2t} \\ -6Ae^{2t} &= 3e^{2t} \\ A &= -\frac{1}{2}, \end{aligned}$$

so $y_{p_1} = -\frac{1}{2}e^{2t}$. *For the second problem, we guess* $y_{p_2} = A\cos(t) + B\sin(t)$, *so*

$$\begin{aligned} y_{p_2}'' - 3y_{p_2}' - 4y_{p_2} &= 2\sin(t) \\ -A\cos(t) - B\sin(t) - 3\left(-A\sin(t) + B\cos(t)\right) - 4\left(A\cos(t) + B\sin(t)\right) &= 2\sin(t) \\ (-5A - 3B)\cos(t) + (-5B + 3A)\sin(t) &= 2\sin(t). \end{aligned}$$

Comparing the coefficients on each side of the equation, we see that we require

$$\begin{aligned} -5A - 3B &= 0 \\ -5B + 3A &= 2, \end{aligned}$$

so $A = -\frac{3}{5}B$. *Using this in the second equation, we find that*

$$\begin{aligned} -5B - \frac{9}{5}B &= 2 \\ B &= -\frac{5}{17}, \end{aligned}$$

so $A = \frac{3}{17}$. *Therefore,*

$$y_{p_2} = \frac{3}{17}\cos t - \frac{5}{17}\sin t.$$

Finally, for the third equation, we guess

$$y_{p_3} = Ae^t\cos(2t) + Be^t\sin(2t).$$

After using the ansatz in the third equation, we find that we require

$$\begin{aligned} -10A - 2B &= -8 \\ -10B + 2A &= 0, \end{aligned}$$

so $A = \frac{10}{13}$ *and* $B = \frac{2}{13}$. *Therefore,*

$$y_{p_3} = \frac{10}{13}e^t\cos(2t) + \frac{2}{13}e^t\sin(2t).$$

Now, putting everything together, our particular solution is

$$y_p = -\frac{1}{2}e^{2t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t + \frac{10}{13}e^t\cos(2t) + \frac{2}{13}e^t\sin(2t).$$

Sometimes, the guess for the form of the particular solution may overlap with the homogeneous solution. In such cases, we use the same idea as when we had repeated roots: multiply our ansatz by t.

Example 2.3. *Let us solve the differential equation*

$$y'' - 3y' - 4y = 2e^{-t}.$$

The characteristic equation is $p(r) = r^2 - 3r - 4$, *so*

$$\begin{aligned} r^2 - 3r - 4 &= 0 \\ (r-4)(r+1) &= 0. \end{aligned}$$

Our roots are $r = 4$ *and* $r = -1$, *so our complementary solution is*

$$y_c = c_1e^{4t} + c_2e^{-t}.$$

For the nonhomogeneous equation, our initial guess would be Ae^{-t}, *but this overlaps with the homogeneous solution so we guess*

$$y_p = Ate^{-t}.$$

Now, we have

$$\begin{aligned} y_p'' - 3y_p' - 4y_p &= 2e^{-t} \\ -2Ae^{-t} + Ate^{-t} - 3\left(Ae^{-t} - Ate^{-t}\right) - 4Ate^{-t} &= 2e^{-t} \\ -5Ae^{-t} &= 2e^{-t}, \end{aligned}$$

so $A = -\frac{2}{5}$. *Thus, the particular solution is*

$$y_p = -\frac{2}{5}e^{-t}.$$

When multiplying by t does not work, repeat the process so that $y_p = t^2e^{rt}$ becomes your ansatz.

3 Variation of Parameters

Consider the second-order differential equation

$$y'' + p(t)y' + q(t)y = g(t),$$

and assume we know the general solution of the homogeneous equation to be

$$y_c = c_1 y_1 + c_2 y_2.$$

Now, we suppose that the particular solution is

$$y_p = u_1(t)y_1 + u_2(t)y_2,$$

where u_1 and u_2 are to-be-determined. Note that we have one equation for two unknown functions, so there may be many choices of u_1 and u_2. We see that

$$y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2',$$

and we add the condition

$$u_1'y_1 + u_2'y_2 = 0,$$

so that

$$y_p' = u_1y_1' + u_2y_2'.$$

Next, we find that

$$y_p'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'',$$

and using y_p in the differential equation, we have

$$\begin{aligned} y_p'' + p(t)y_p' + q(t)y_p &= u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' + p(t)\left(u_1y_1' + u_2y_2'\right) + q(t)\left(u_1(t)y_1 + u_2(t)y_2\right) \\ &= u_1'y_1' + u_2'y_2' + u_1\left(y_1'' + p(t)y_1' + q(t)y_1\right) + u_2\left(y_2'' + p(t)y_2' + q(t)y_2\right) \\ &= u_1'y_1' + u_2'y_2', \end{aligned}$$

since y_1 and y_2 are solutions to the homogeneous equation. Therefore, we require

$$u_1'y_1' + u_2'y_2' = g$$

Now, we have the system of equations

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= g(t), \end{aligned}$$

or in matrix form

$$\begin{aligned} \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} &= \begin{bmatrix} 0 \\ g(t) \end{bmatrix} \\ \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} &= \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ g(t) \end{bmatrix} \\ &= \frac{1}{y_1y_2' - y_2y_1'} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ g(t) \end{bmatrix} \\ &= \frac{1}{W[y_1, y_2](t)} \begin{bmatrix} -y_2g(t) \\ y_1g(t) \end{bmatrix}, \end{aligned}$$

and so

$$\begin{aligned} u_1' &= \frac{1}{W[y_1, y_2](t)}(-y_2g(t)) \\ u_2' &= \frac{1}{W[y_1, y_2](t)}y_2g(t). \end{aligned}$$

Thus,

$$u_1 = \int -\frac{y_2 g(t)}{W[y_1, y_2](t)}\, dt + c_3,$$

and

$$u_2 = \int \frac{y_1 g(t)}{W[y_1, y_2](t)}\, dt + c_4.$$

Therefore, our general solution is

$$\begin{aligned} y &= y_c + y_p \\ &= c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2 \\ &= c_1 y_1 + c_2 y_2 + \left(\int -\frac{y_2 g(t)}{W[y_1, y_2](t)}\, dt + c_3\right) y_1 + \left(\int \frac{y_1 g(t)}{W[y_1, y_2](t)}\, dt + c_4\right) y_2 \\ &= c_5 y_1 + c_6 y_2 - y_1 \int \frac{y_2 g(t)}{W[y_1, y_2](t)}\, dt + y_2 \int \frac{y_1 g(t)}{W[y_1, y_2](t)}\, dt. \end{aligned}$$

Example 3.1. *Find the general solution of*

$$y'' + 4y = 8\tan(t), \quad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$