1. Using the appropriate theorem for first order differential equations, determine the region in which a unique solution of the differential equation exists.

(a) \( \frac{dy}{dt} + \tan(t)y = \sin(t) \)

(b) \( \frac{dy}{dt} = \sqrt{1 - t^2 - y^2} \)

(c) \( \frac{dy}{dt} = \frac{1 + t^2}{3y - y^2} \)

2. Consider the first order nonlinear differential equation of the form

\[ y' + p(t)y = q(t)y^n \]

called a Bernoulli equation.

(a) Solve the equation for \( n = 0, n = 1 \).

(b) Consider the substitution \( v = y^{1-n} \). Transform the differential equation into an equation involving \( v \) and \( t \).

(c) Solve the transformed differential equation.

(d) Use the method you have developed to solve

\[ y' = \epsilon y - \sigma y^3 \]

where \( \epsilon > 0 \) and \( \sigma > 0 \).

3. Consider the initial value problem

\[ y' + p(t)y = g(t), \quad y(t_0) = y_0 \]

where

\[ g(t) = \begin{cases} g_1, & t_0 \leq t \leq t_1, \\ g_2, & t > t_1 \end{cases} \]

and

\[ p(t) = \begin{cases} p_1, & t_0 \leq t \leq t_2, \\ p_2, & t > t_2. \end{cases} \]

Take \( g_1, g_2, p_1, \) and \( p_2 \) to be constants, and assume \( t_1 < t_2 \). Solve the differential equation by solving it separately over intervals where \( g(t) \) and \( p(t) \) are continuous.

4. Consider the autonomous differential equation

\[ \frac{dy}{dt} = f(y). \]

For the following, sketch \( f(y) \) vs \( y \), determine and classify equilibria, draw the phase line, and sketch solution trajectories in the \( ty \)-plane.

(a) \( f(y) = y^2(y^2 - 1) \)

(b) \( f(y) = y^2(4 - y^2) \)
5. The Monterey Bay was one of the world’s most productive sardine fisheries until the fishing industry collapsed in the mid-1950’s due overfishing and other factors. To prevent this from happening to other fisheries, let’s explore the Schaefer model

\[ \frac{dy}{dt} = r \left(1 - \frac{y}{K}\right) y - Ey \]

and develop a sustainable harvest plan for the fishery. Here \( y \) is the fish population, \( Ey \) respresents the rate at which fish are harvested, and \( E \) represents the effort expended in harvesting fish.

(a) Show that if \( E < r \), then there are two equilibrium points and classify the equilibria.
(b) A sustainable yield \( Y \) is the rate at which fish may be caught indefinitely. Determine \( Y \) as a function of \( E \).
(c) Determine the value of \( E \) which maximizes \( Y \). This maximum is called the maximum sustainable yield.

6. Consider the following initial value problem

\[ y' = 0.5 - t + 2y, \quad y(0) = 1 \]

(a) Use Euler’s method with \( h = 0.1 \) to approximate the solution for \( t \in [0, 0.4] \).
(b) Use implicit Euler’s method with \( h = 0.1 \) to approximate the solution for \( t \in [0, 0.4] \).
(c) Use Euler’s method with \( h = 0.025 \) to approximate the solution for \( t \in [0, 0.4] \).
(d) Solve the initial value problem to find \( y(t) \). Compare the values of the solution to the approximate values you found in parts (a), (b), and (c).

7. Consider the initial value problem

\[ y' = 1 - t + y, \quad y(t_0) = y_0. \]

(a) Find the exact solution.
(b) Using the Euler scheme, show that

\[ y_k = (1 + h)y_{k-1} + h - ht_{k-1} \]

for \( k = 1, 2, \ldots \).
(c) Inductively show that

\[ y_n = (1 + h)^n(y_0 - t_0) + t_n. \]
(d) Show that as \( n \to \infty \), \( y_n \to y(t) \). That is the Euler scheme converges to the exact solution. Recall the limit

\[ \lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^n = e^a. \]