# MAT 22B Problem Set 3 (Due 8/17)

1. Using the appropriate theorem for first order differential equations, determine the region in which a unique solution of the differential equation exists.

   (a) $\frac{dy}{dt} + \tan(t)y = \sin(t)$

   (b) $\frac{dy}{dt} = \sqrt{1 - t^2 - y^2}$

   (c) $\frac{dy}{dt} = \frac{1+t^2}{3y-y^2}$

2. Consider the first order nonlinear differential equation of the form

$$y' + p(t)y = q(t)y^n$$

   called a Bernoulli equation.

   (a) Solve the equation for $n = 0$, $n = 1$.

   (b) Consider the substitution $v = y^{1-n}$. Transform the differential equation into an equation involving $v$ and $t$.

   (c) Solve the transformed differential equation.

   (d) Use the method you have developed to solve

$$y' = \epsilon y - \sigma y^3$$

   where $\epsilon > 0$ and $\sigma > 0$.

3. Consider the initial value problem

$$y' + p(t)y = g(t), \quad y(t_0) = y_0$$

   where

$$g(t) = \begin{cases} g_1, & t_0 \leq t \leq t_1, \\ g_2, & t > t_1 \end{cases}$$

   and

$$p(t) = \begin{cases} p_1, & t_0 \leq t \leq t_2, \\ p_2, & t > t_2. \end{cases}$$

   Take $g_1$, $g_2$, $p_1$, and $p_2$ to be constants, and assume $t_1 < t_2$. Solve the differential equation by solving it separately over intervals where $g(t)$ and $p(t)$ are continuous.

4. Consider the autonomous differential equation

$$\frac{dy}{dt} = f(y).$$

   For the following, sketch $f(y)$ vs $y$, determine and classify equilibria, draw the phase line, and sketch solution trajectories in the $ty$-plane.

   (a) $f(y) = y^2(y^2 - 1)$

   (b) $f(y) = y^2(4 - y^2)$

5. The Monterey Bay was one of the world's most productive sardine fisheries until the fishing industry collapsed in the mid-1950's due overfishing and other factors. To prevent this from happening to other fisheries, let's explore the Schafer model
$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y - Ey$$
and develop a sustainable harvest plan for the fishery. Here $y$ is the fish population, $Ey$ resprents the rate at which fish are harvested, and $E$ represents the effort expended in harvesting fish.

   (a) Show that if $E < r$, then there are two equilibrium points and classify the equilibria.

   (b) A sustainable yield $Y$ is the rate at which fish may be caught indefinitely. Determine $Y$ as a function of $E$.

   (c) Determine the value of $E$ which maximizes $Y$. This maximum is called the maximum sustainable yield.

6. Consider the following initial value problem
$$y' = 0.5 - t + 2y, \quad y(0) = 1$$

   (a) Use Euler's method with $h = 0.1$ to approximate the solution for $t \in [0, 0.4]$.

   (b) Use implicit Euler's method with $h = 0.1$ to approximate the solution for $t \in [0, 0.4]$.

   (c) Use Euler's method with $h = 0.025$ to approximate the solution for $t \in [0, 0.4]$.

   (d) Solve the initial value problem to find $y(t)$. Compare the values of the solution to the approximate values you found in parts (a), (b), and (c).

7. Consider the initial value problem
$$y' = 1 - t + y, \quad y(t_0) = y_0.$$

   (a) Find the exact solution.

   (b) Using the Euler scheme, show that
$$y_k = (1+h)y_{k-1} + h - ht_{k-1}$$
   for $k = 1, 2, \ldots$.

   (c) Inductively show that
$$y_n = (1+h)^n(y_0 - t_0) + t_n.$$

   (d) Show that as $n \to \infty$, $y_n \to y(t)$. That is the Euler scheme converges to the exact solution. Recall the limit $\lim_{n\to\infty}\left(1 + \frac{a}{n}\right)^n = e^a$.