1. Consider the following initial value problem

\[ y' = (t - 1)^2(y - 2) - t + 1, \quad y(1) = 2. \]

(a) Transform the initial value problem so that the initial point is at the origin.
(b) Let \( \phi_n(t) \) be the iterates from the method of successive approximations. Assume \( \phi_0(t) = 0 \). Find \( \phi_n(t) \) for arbitrary values of \( n \).
(c) Plot \( \phi_n(t) \) for \( n = 1, 2, 3, 4 \). Determine whether the approximates appear to be converging.
(d) Show that the sequence \( \{\phi_n(t)\} \) converges.

2. Consider the initial value problem

\[ y' = f(t, y), \quad y(0) = 0. \]

(a) Show that if \( \frac{\partial f}{\partial y} \) is continuous in a rectangle \( D \), then there is a constant \( K > 0 \) such that

\[ |f(t, y_1) - f(t, y_2)| \leq K |y_1 - y_2| \]

for any \( (t, y_1), (t, y_2) \in R \).
(b) Let \( \phi_n, \phi_{n-1} \) be two iterates from the method of successive approximations. Show that

\[ |f(t, \phi_n(t)) - f(t, \phi_{n-1}(t))| \leq K|\phi_n(t) - \phi_{n-1}(t)|. \]
(c) Suppose

\[ \phi(t) = \int_0^t f(s, \phi(s)) \, ds \]

and \( \psi(t) \) is another solution to the integral equation. Show that it must be the case that \( \phi(t) = \psi(t) \). That is, show that the solution to the integral equation is unique.

3. Solve the following finite difference equations in terms of an initial value \( y_0 \) and determine the behavior of the solution as \( n \to \infty \).

(a) \[ y_{n+1} = \sqrt{\frac{n+3}{n+1}} y_n \]
(b) \[ y_{n+1} = 0.5 y_n + 6 \]

4. Suppose you invest $1000 in the SPY ETF and you expect that the investment will return 8% compounded monthly, and suppose you continue investing $25 a month. How much do you expect your investment to be worth in 10 years?

5. Solve the following initial value problems and describe the behavior of the solution as \( t \to \infty \).

(a) \[ y'' + 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = -1 \]
(b) \[ 2y'' + y' - 4y = 0, \quad y(0) = 0, \quad y'(0) = 1 \]
(c) \[ 4y'' - y = 0, \quad y(-2) = 1, \quad y'(-2) = -1 \]

6. Consider the differential equation

\[ ay'' + by' + cy = 0 \]

where \( a, b, \) and \( c \) are constants and \( a > 0 \). Determine conditions on \( a, b, \) and \( c \) so that the roots of the characteristic equation are:
(a) distinct and positive.
(b) distinct and negative.
(c) opposite signs.

For each case determine the behavior of the solution as $t \to \infty$.

7. Verify that $y_1$ and $y_2$ are solutions of the differential equation, and determine whether they constitute a fundamental set of solutions.

$$y'' - 2y' + y = 0, \quad y_1(t) = e^t, \quad y_2(t) = te^t.$$  

8. Find the Wronskian of the two solutions of the differential equation

$$\cos(t)y'' + \sin(t)y' - ty = 0.$$  

9. Prove the following statements:

(a) If $y_1$ and $y_2$ are zero at the same point in $I$, then they cannot be a fundamental set of solutions on $I$.

(b) If $y_1$ and $y_2$ have a common point of inflection $t_0$ in $I$, then they cannot be a fundamental set of solutions on $I$ unless both $p$ and $q$ are zero at $t_0$.

10. Solve the initial value problems and determine the behavior of the solution as $t \to \infty$.

(a) $y'' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$

(b) $y'' + y = 0, \quad y\left(\frac{\pi}{3}\right) = 2, \quad y'\left(\frac{\pi}{3}\right) = -4$

(c) $3u'' - u' + 2u = 0, \quad u(0) = 2, \quad u'(0) = 0$

11. An equation of the form

$$t^2 \frac{d^2y}{dt^2} + \alpha t \frac{dy}{dt} + \beta y = 0, \quad t > 0$$

where $\alpha$ and $\beta$ are real constants is called an Euler equation.

(a) Using the change of variable $x = \ln t$, transform the differential equation into a constant coefficient differential equation.

(b) Use this change of variable to solve $t^2y'' + 3ty' - 3y = 0$. 
