MAT 22B Problem Set 4 (Due 8/22)

1. Consider the following initial value problem
$$y' = (t-1)^2(y-2) - t + 1, \quad y(1) = 2.$$
   (a) Transform the initial value problem so that the initial point is at the origin.
   (b) Let $\phi_n(t)$ be the iterates from the method of successive approximations. Assume $\phi_0(t) = 0$. Find $\phi_n(t)$ for arbitary values of n.
   (c) Plot $\phi_n(t)$ for $n = 1, 2, 3, 4$. Determine whether the approximates appear to be converging.
   (d) Show that the sequence $\{\phi_n(t)\}$ converges.

2. Consider the initial value problem
$$y' = f(t, y), \quad y(0) = 0.$$
   (a) Show that if $\frac{\partial f}{\partial y}$ is continuous in a rectangle D, then there is a constant $K > 0$ such that
$$|f(t, y_1) - f(t, y_2)| \le K |y_1 - y_2|$$
   for any $(t, y_1), (t, y_2) \in R$.
   (b) Let ϕ_n, ϕ_{n-1} be two iterates from the method of successive approximations. Show that
$$|f(t, \phi_n(t)) - f(t, \phi_{n-1}(t)| \le K |\phi_n(t) - \phi_{n-1}(t)| .$$
   (c) Suppose
$$\phi(t) = \int_0^t f(s, \phi(s))\, ds$$
   and $\psi(t)$ is another solution to the integral equation. Show that it must be the case that $\phi(t) = \psi(t)$. That is, show that the solution to the integral equation is unique.

3. Solve the following finite difference equations in terms of an initial value y_0 and determine the behavior of the solution as $n \to \infty$.
   (a) $y_{n+1} = \sqrt{\frac{n+3}{n+1}} y_n$
   (b) $y_{n+1} = 0.5 y_n + 6$

4. Suppose you invest \$1000 in the SPY ETF and you expect that the investment will return 8% compounded monthly, and suppose you continue investing \$25 a month. How much do you expect your investment to be worth in 10 years?

5. Solve the following initial value problems and describe the behavior of the solution as $t \to \infty$
   (a) $y'' + 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = -1$
   (b) $2y'' + y' - 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$
   (c) $4y'' - y = 0, \quad y(-2) = 1, \quad y'(-2) = -1$

6. Consider the differential equation
$$ay'' + by' + cy = 0$$
   where a, b, and c are constants and $a > 0$. Determine conditions on a, b, and c so that the roots of the characteristic equation are:

(a) distinct and positive.
(b) distinct and negative.
(c) opposite signs.

For each case determine the behavior of the solution as $t \to \infty$.

7. Verify that y_1 and y_2 are solutions of the differential equation, and determine whether they constitute a fundamental set of solutions.
$$y'' - 2y' + y = 0, \quad y_1(t) = e^t, \quad y_2(t) = te^t.$$

8. Find the Wronskian of the two solutions of the differential equation
$$\cos(t)y'' + \sin(t)y' - ty = 0.$$

9. Prove the following statements:
   (a) If y_1 and y_2 are zero at the same point in I, then they cannot be a fundamental set of solutions on I.
   (b) If y_1 and y_2 have a common point of inflection t_0 in I, then they cannot be a fundamental set of solutions on I unless both p and q are zero at t_0.

10. Solve the initial value problems and determine the behavior of the solution as $t \to \infty$.
   (a) $y'' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$
   (b) $y'' + y = 0, \quad y\left(\frac{\pi}{3}\right) = 2, \quad y'\left(\frac{\pi}{3}\right) = -4$
   (c) $3u'' - u' + 2u = 0, \quad u(0) = 2, \quad u'(0) = 0$

11. An equation of the form
$$t^2 \frac{d^2y}{dt^2} + \alpha t \frac{dy}{dt} + \beta y = 0, \quad t > 0$$
where α and β are real constants is called an Euler equation.
   (a) Using the change of variable $x = \ln t$, transform the differential equation into a constant coefficient differential equation.
   (b) Use this change of variable to solve $t^2y'' + 3ty' - 3y = 0$.