Fast Interface Models with Dimensional Reduction

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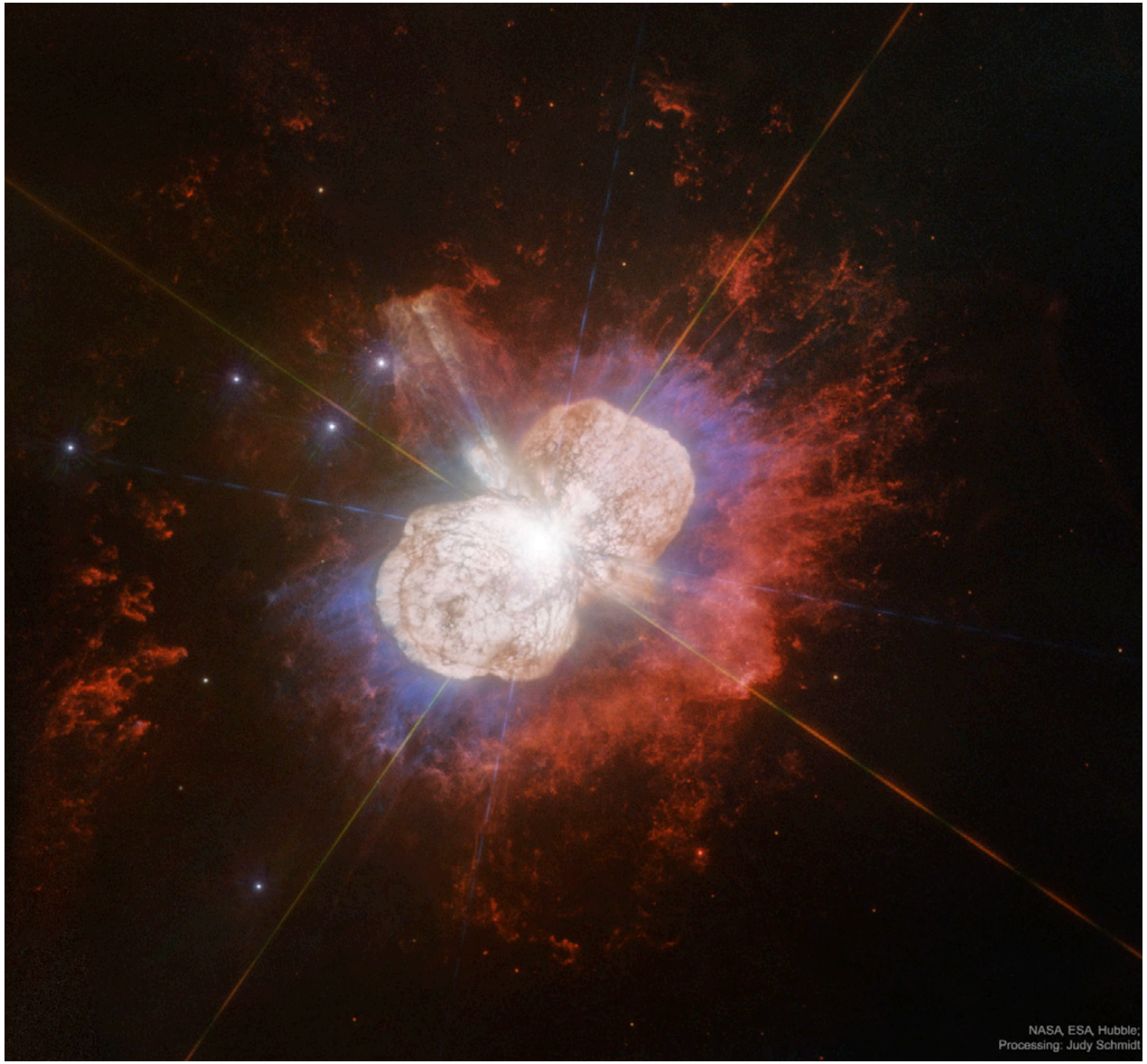
Methodology

Multiscale Flow via Interface Models

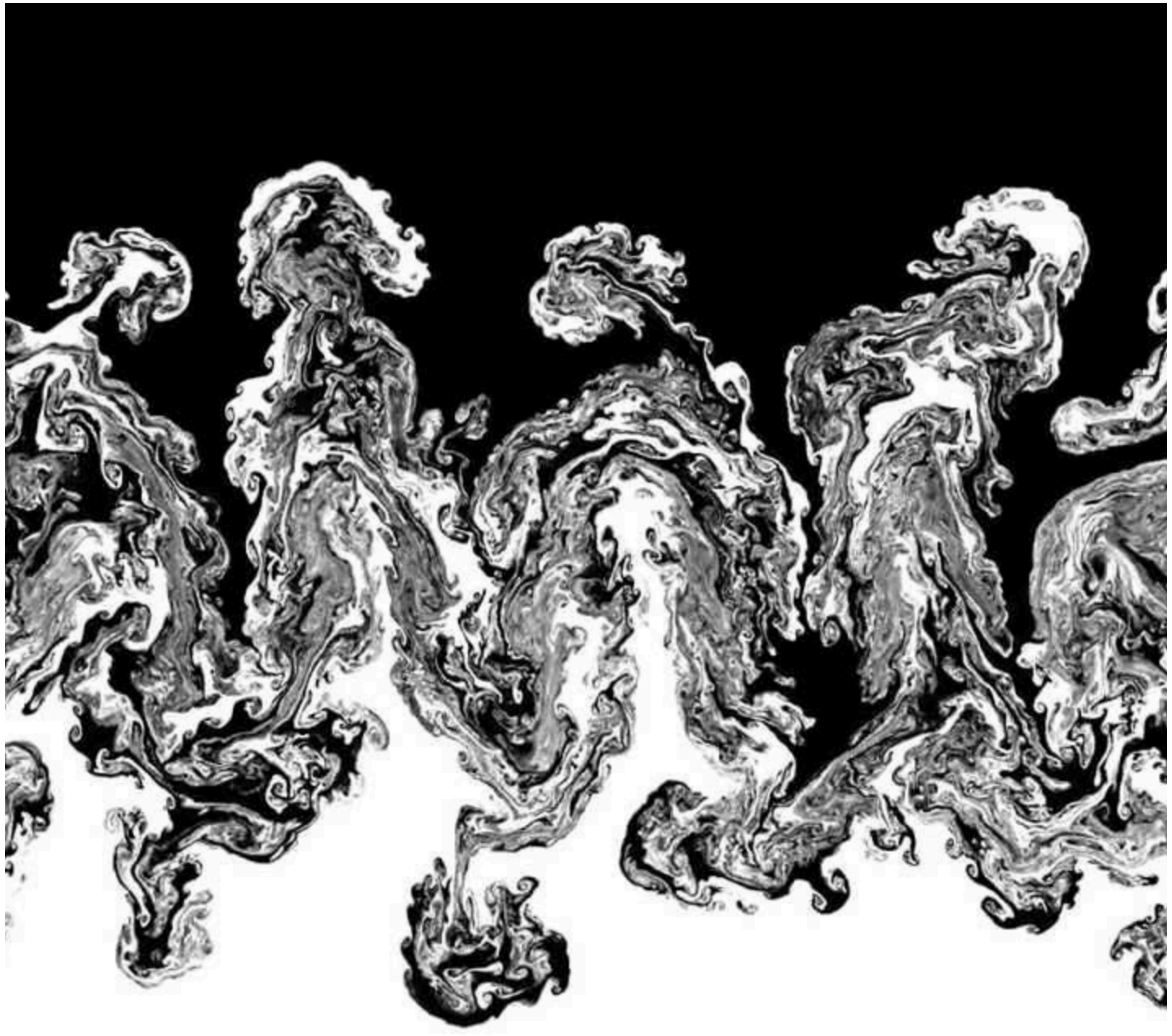
- Action at many scales
- Direct numerical simulation is computationally expensive
- **Idea:** decompose flow into simple constituent parts
    - e.g. transport, diffusion, shock waves, contacts, acoustic waves, flame fronts, etc.
    - Model using interface methods



Interface Methods

Strengths and Weaknesses

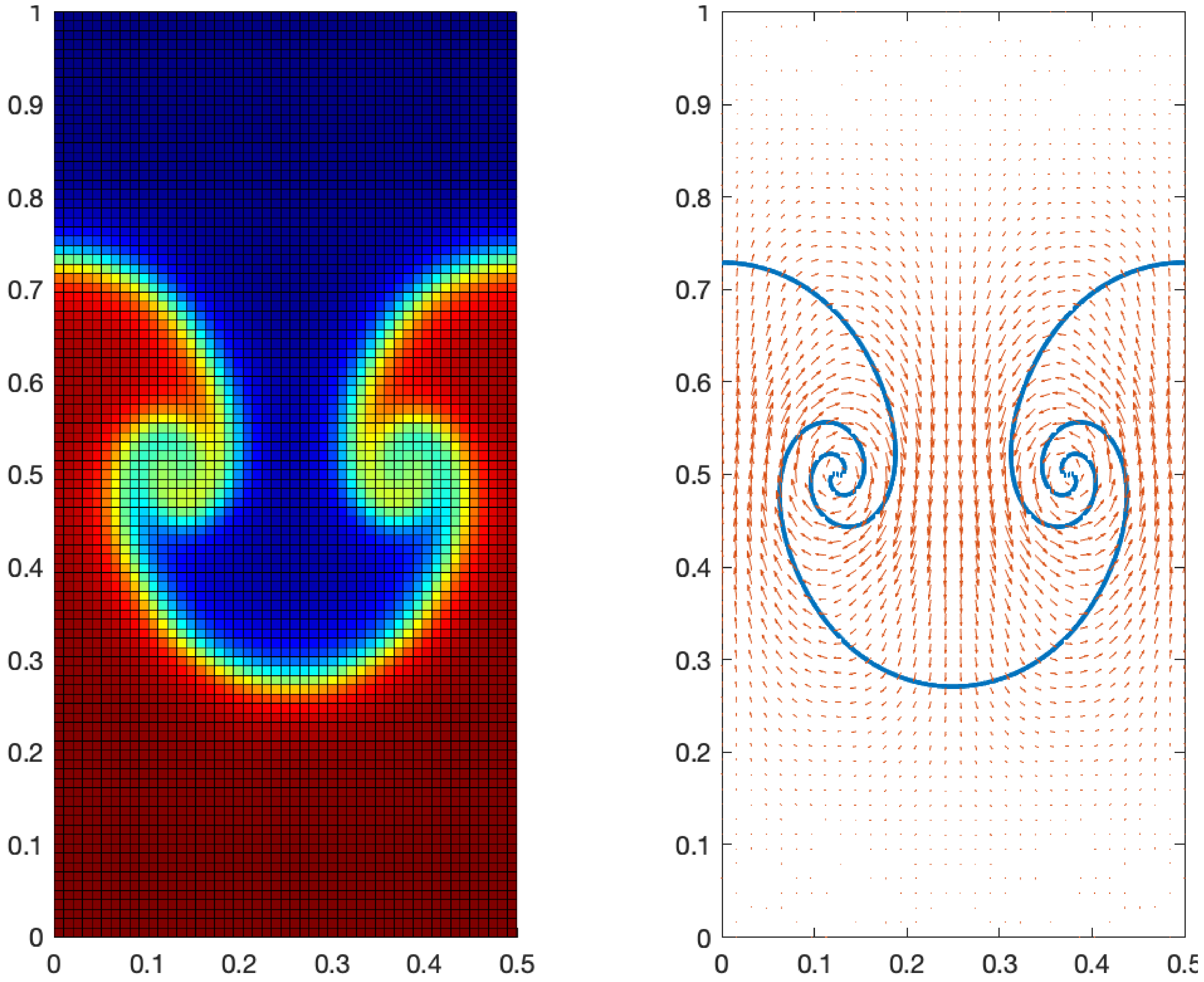
- Fluid systems often dominated by propagating waves and surfaces of discontinuity.
- Interface methods offer geometric flexibility, computational efficiency, and concise representation.
- Less useful at scales where flow is dominated by mixing or turbulence.



Incompressible Flow

Contact Discontinuities

- Nonlocal effects of incompressible flow recovered using boundary integral methods.
- We developed the **z-model** for contact discontinuities in 2d & 3d.
- Matches analytical and experimental predictions for Rayleigh-Taylor Instability (RTI).

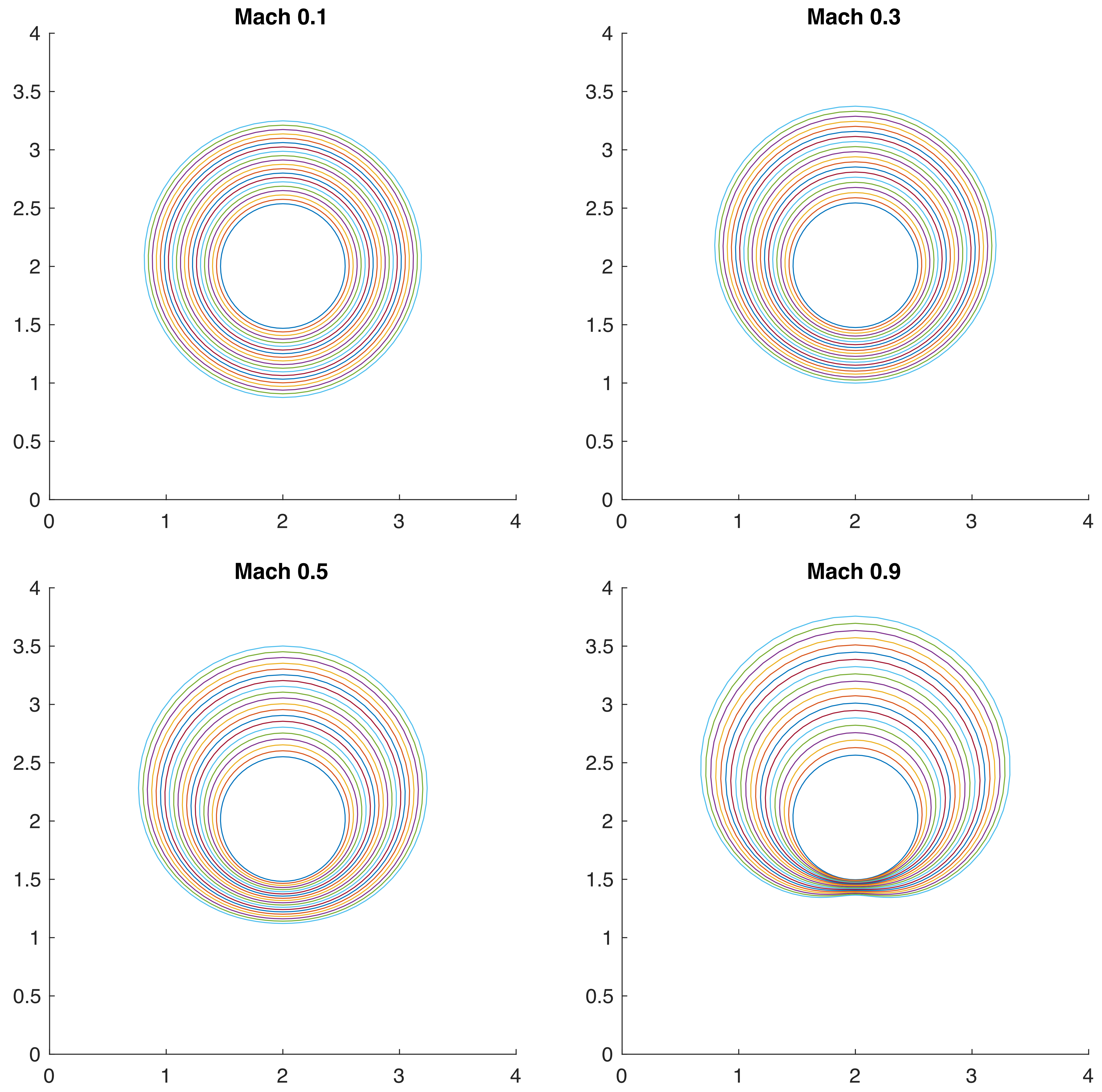


Rayleigh-Taylor instability (z-model with reconstructed density/velocity)

Compressible Flow

Nonlinear Sound Waves

- Novel geometric coordinates which align with propagating wave or surface of discontinuity.
- The **wza-model** for nonlinear sound waves in 2d.
- Extension to shock waves/vorticity waves/contacts in progress



Sound wave emitted from moving object at various speeds (abc-model)

Relevant Publications

1. **Computational Methods for the Euler Equations of Gas Dynamics**
    1. J. REISNER, J. SERENCSA, AND S. SHKOLLER, **A space-time smooth artificial viscosity method for nonlinear conservation laws**, *Journal of Computational Physics, 235,* 912–933, (2013)
    2. R. RAMANI, J. REISNER, AND S. SHKOLLER, **A space-time smooth artificial viscosity method with wavelet noise indicator and shock collision scheme, Part 1: the 1-D case**, *Journal of Computational Physics, 387,* 81–116, (2019)
    3. R. RAMANI, J. REISNER, AND S. SHKOLLER, **A space-time smooth artificial viscosity method with wavelet noise indicator and shock collision scheme, Part 2: the 2-D case**, *Journal of Computational Physics, 387*, 45–80, (2019)
    4. R. RAMANI and S. SHKOLLER, **A multiscale model for Rayleigh-Taylor and Richtmyer-Meshkov instabilities**, *Journal of Computational Physics, 405*, 109177, (2020)
2. **Fast interface Models for Contact Discontinuities in Incompressible Flow**
    1. A. CHENG, R. GRANERO-BELINCHON, S. SHKOLLER, and J. WILKENING, **Rigorous asymptotic models of water waves**, *Water Waves, 1* , 71--130, (2019)
    2. J. CANFIELD, N. DENISSEN, M. FRANCOIS, R. GORE, R. RAUENZAHN, J. REISNER, S. SHKOLLER, **A comparison of interface growth models applied to Rayleigh-Taylor and Richtmyer-Meshkov instabilities**, *Journal of Fluids Engineering, 142(12),* 121108, (2020)
    3. G. PANDYA and S. SHKOLLER, **Interface models for three-dimensional Rayleigh-Taylor instability**, *Journal of Fluid Mechanics, 959*, A10, (2023)
3. **Fast Interface models for Sound Waves, Shock Waves, Vorticity Waves, and Contacts in Compressible Flow**
    1. S. SHKOLLER and V. VICOL, **The geometry of maximal development for the Euler equations**, *(2023), preprint*.
    2. G. PANDYA and S. SHKOLLER, **Interface model for nonlinear sound wave propagation based on new geometric Riemann variables**, *In Preparation*.

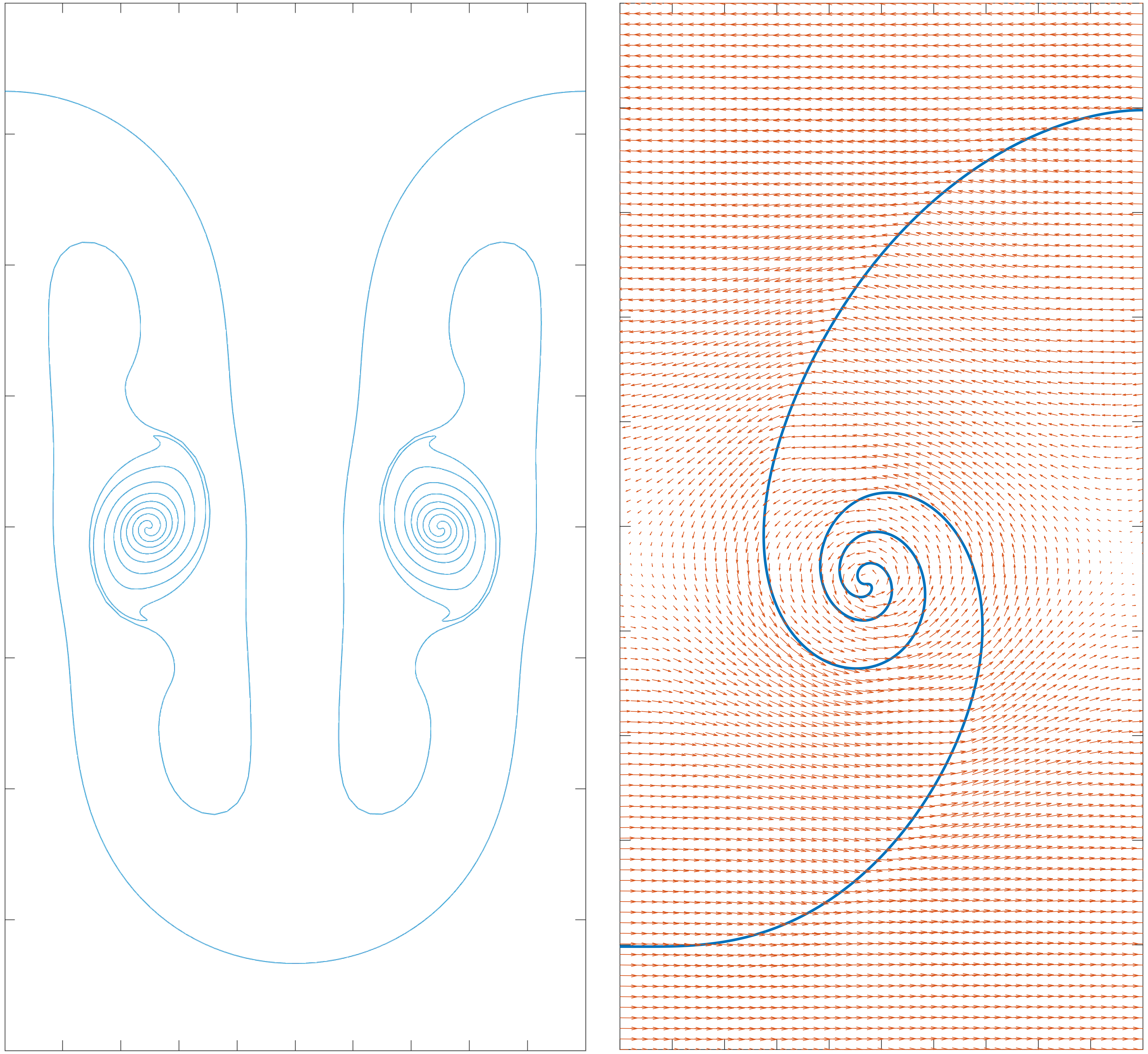
The z-Model

(P. and Shkoller, JFM ’23)

- Interface model for the interface position and amplitude of vorticity.
- Models the deposition of vorticity on density discontinuities:

$$\frac{D\omega}{Dt} = \frac{\nabla\rho \times \nabla p}{\rho^2}.$$

- Velocity reconstructed using nonlocal Biot-Savart integral.



z-model interface, Rayleigh-Taylor instability

with reconstructed velocity

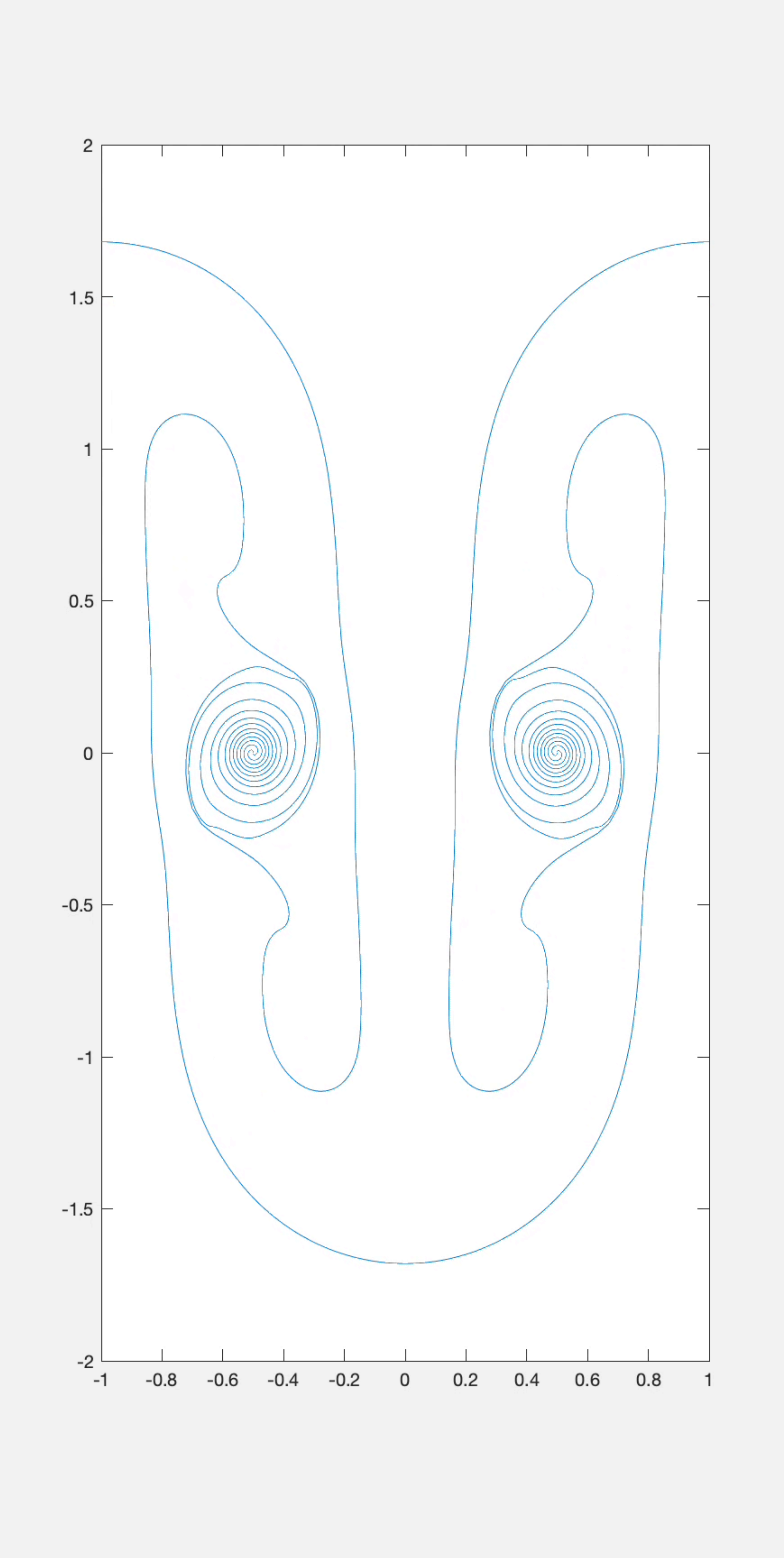
Velocity Reconstruction

- An arbitrary velocity field u can be reconstructed from its **vorticity** $\omega = \operatorname{curl} u$ and **compression** $\chi = \operatorname{div} u$:

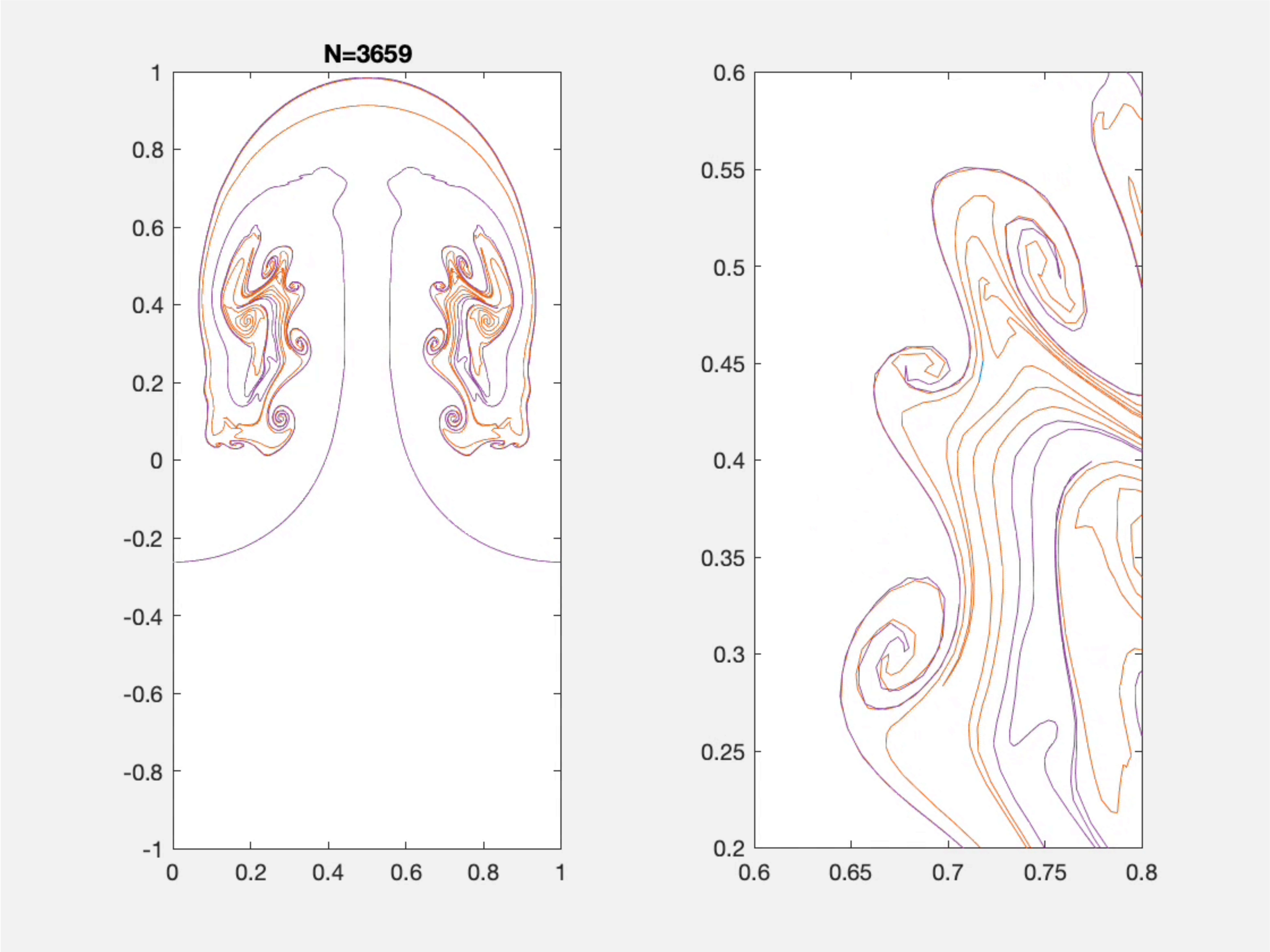
$$u(x) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{\omega(x') \times (x - x')}{|x - x'|^3} + \frac{\chi(x')(x - x')}{|x - x'|^3} dx'$$

- Given a vortex sheet at $\Gamma(t)$ in incompressible flow with amplitude of vorticity $\bar{\omega}(x, t), x \in \Gamma(t)$, the velocity is given by the *Biot-Savart law,*

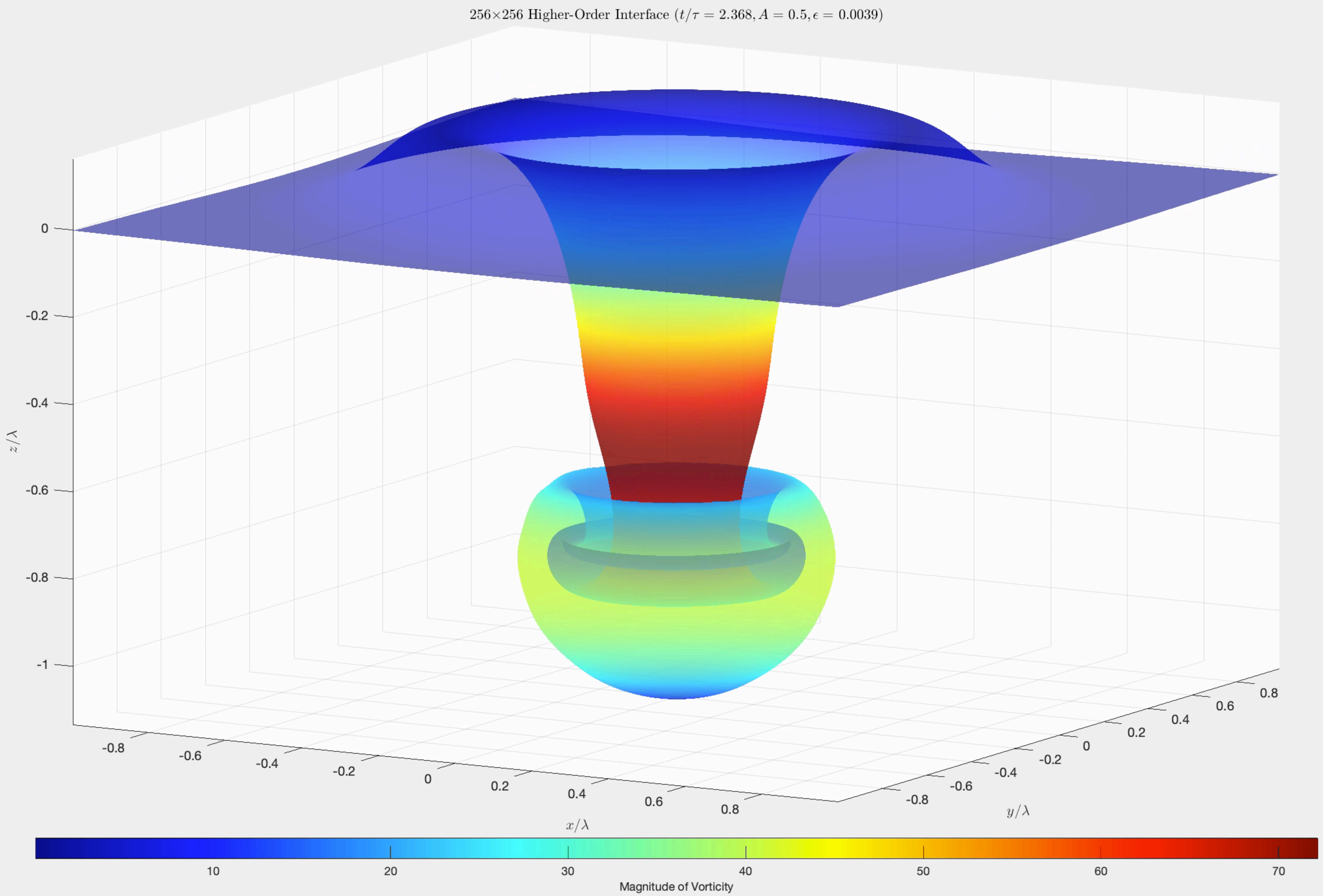
$$u(x, t) = \frac{1}{4\pi} \iint_{\Gamma} \frac{\bar{\omega}(x', t) \times (x - x')}{|x - x'|^3} dS(x')$$



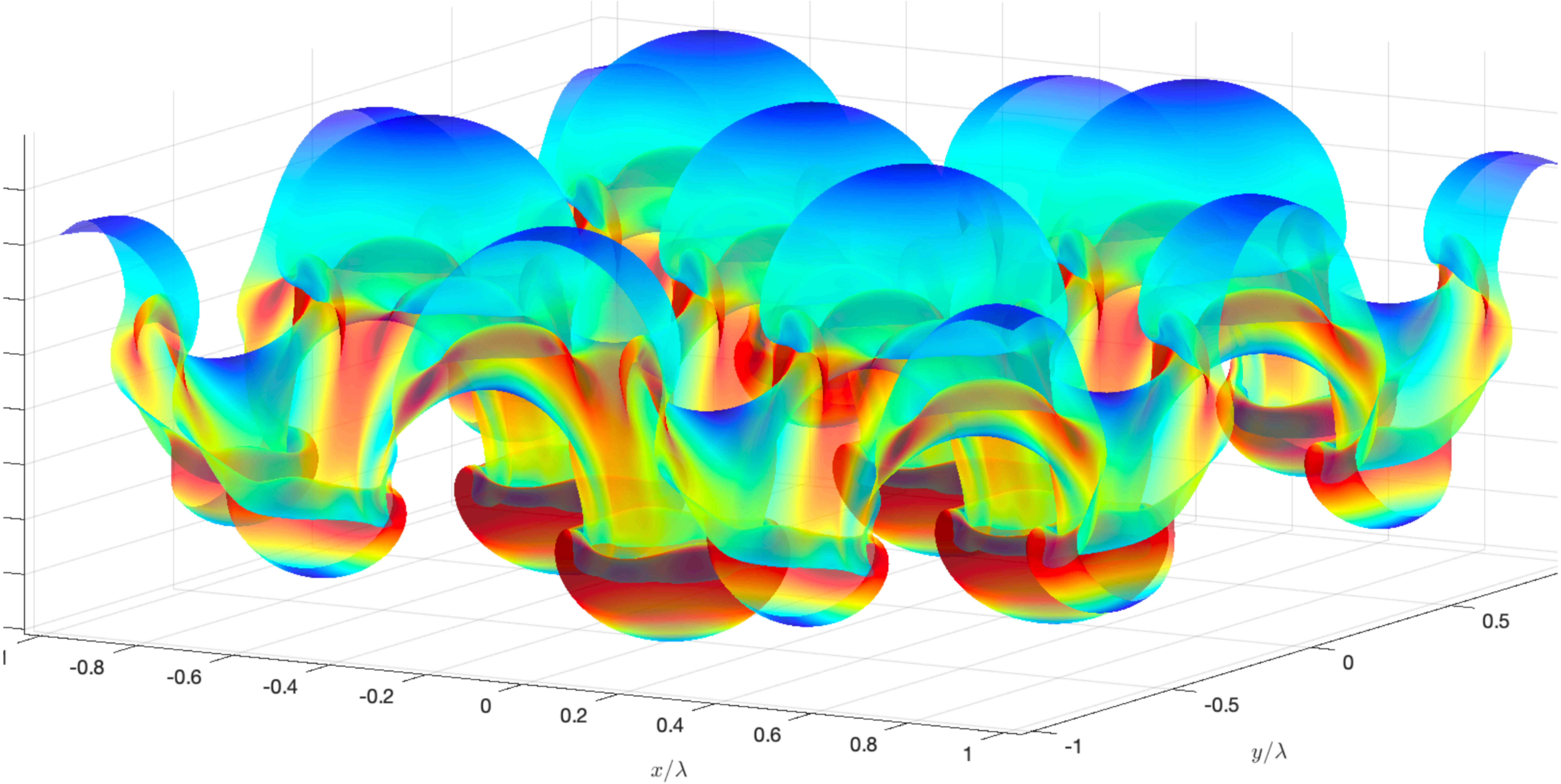
Single-mode Rayleigh-Taylor instability



Rising bubble interacting with unstable density interface


256×256 Higher-Order Interface ($t/\tau = 2.368, A = 0.5, \epsilon = 0.0039$)
z/λ
0
-0.2
-0.4
-0.6
-0.8
-1
x/λ
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
y/λ
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
10
20
30
40
50
60
70
Magnitude of Vorticity

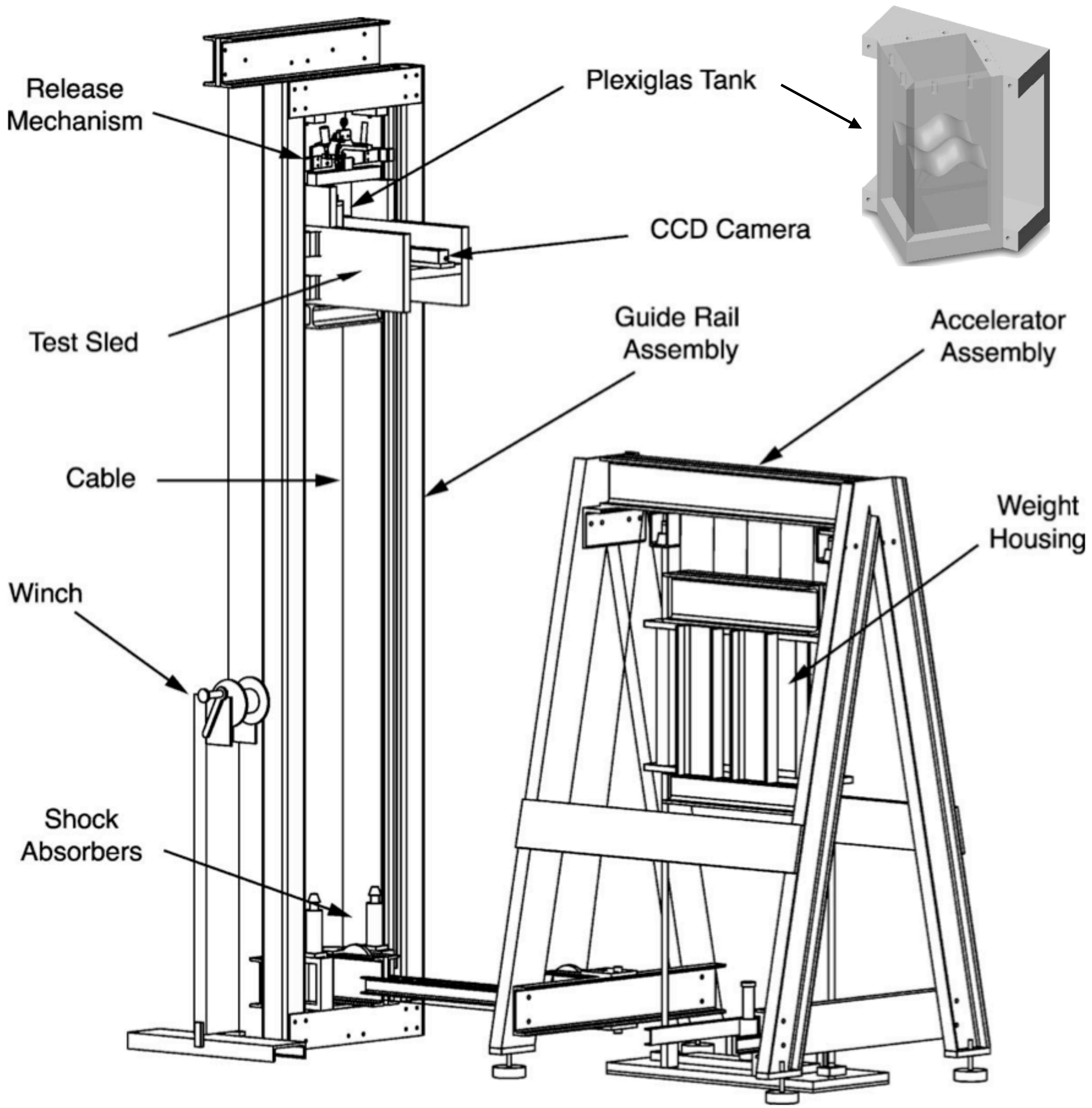
3D Single-Mode RTI



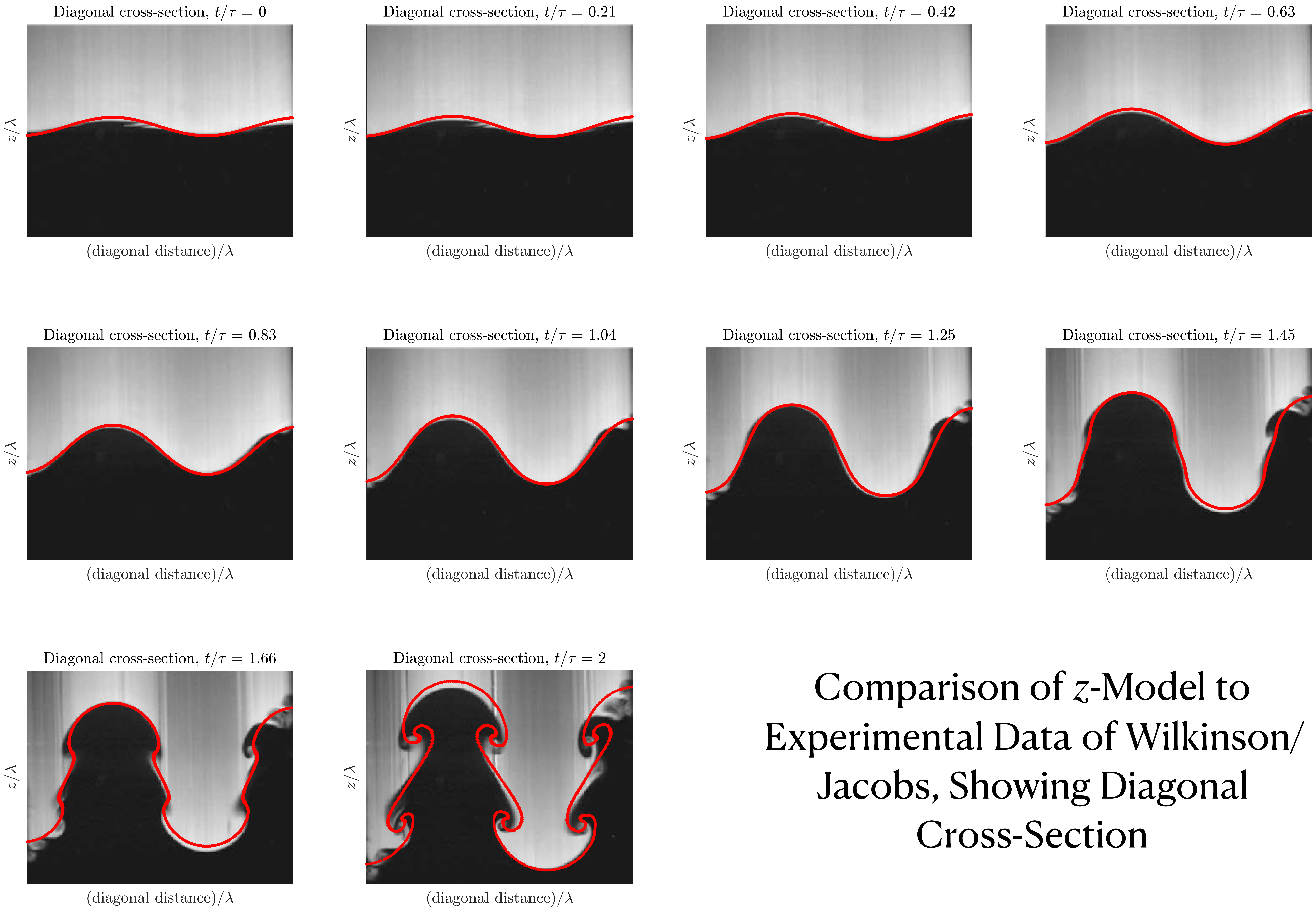
Comparison With Experiments

Single-Mode RT Instability

- A standing wave is excited in a tank containing two fluids, the heavier fluid on the bottom.
- The tank is then accelerated downward, so that the acceleration points from the heavier fluid into the lighter fluid.
- The resulting Rayleigh-Taylor instability is photographed along the tank's diagonal using PLIF (planar laser induced fluorescence).



Experimental setup of Waddell/Jacobs and Wilkinson/Jacobs



Comparison of *z*-Model to Experimental Data of Wilkinson/Jacobs, Showing Diagonal Cross-Section

Comparison With Experiments

Rocket Rig Experiment

- Theory predicts that the RT mixing layer grows quadratically with time:

$$h - h_0 = \alpha A g t^2$$

- Experiments of Read and Youngs found constant of proportionality

$$\alpha \approx 0.06 \text{ or } 0.07$$

- Model of Cabot/Cook & Ristorcelli/Clark measures α dynamically:

$$\alpha = \dot{h}^2/(4Agh)$$

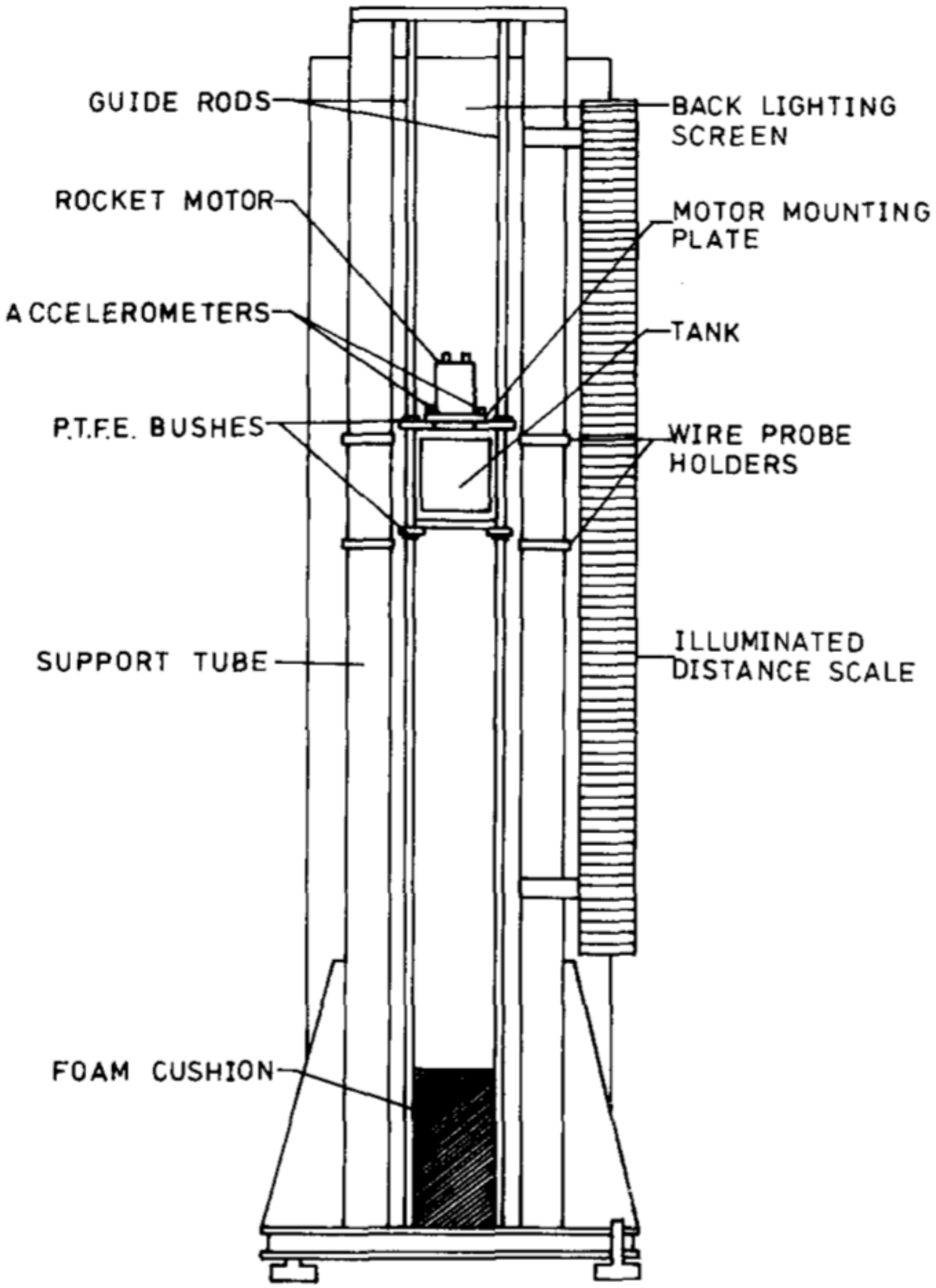
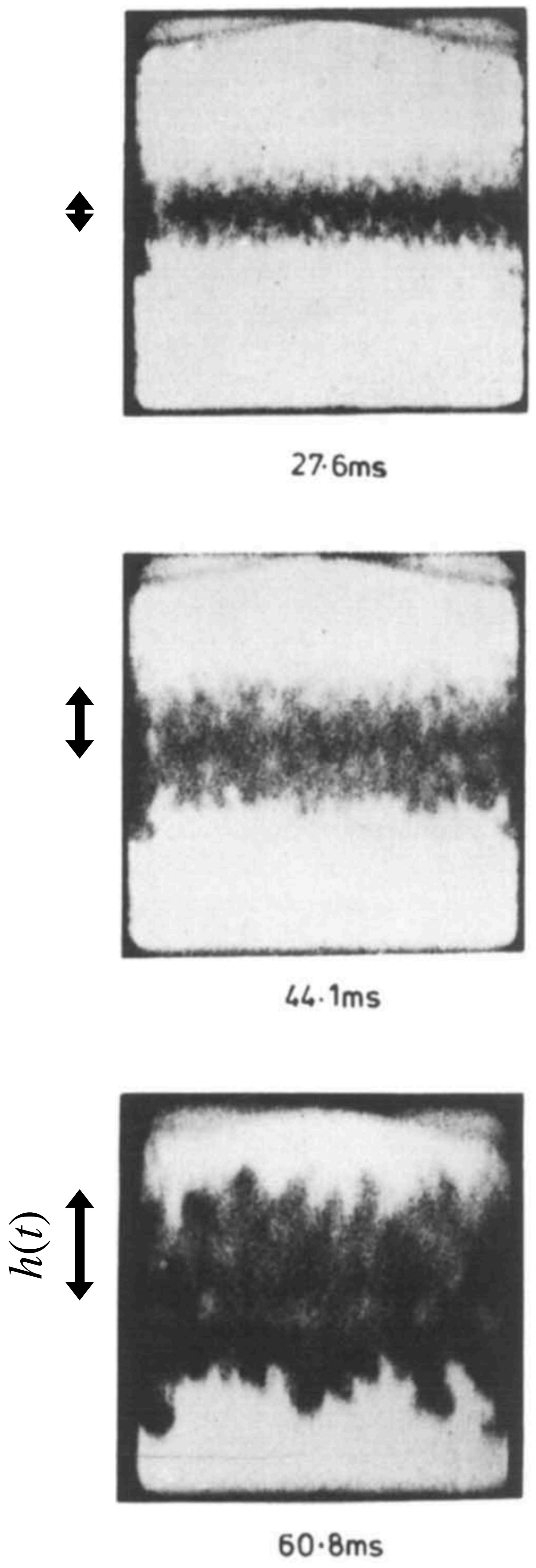


Fig. 1. Experimental apparatus.



Comparison With Experiments

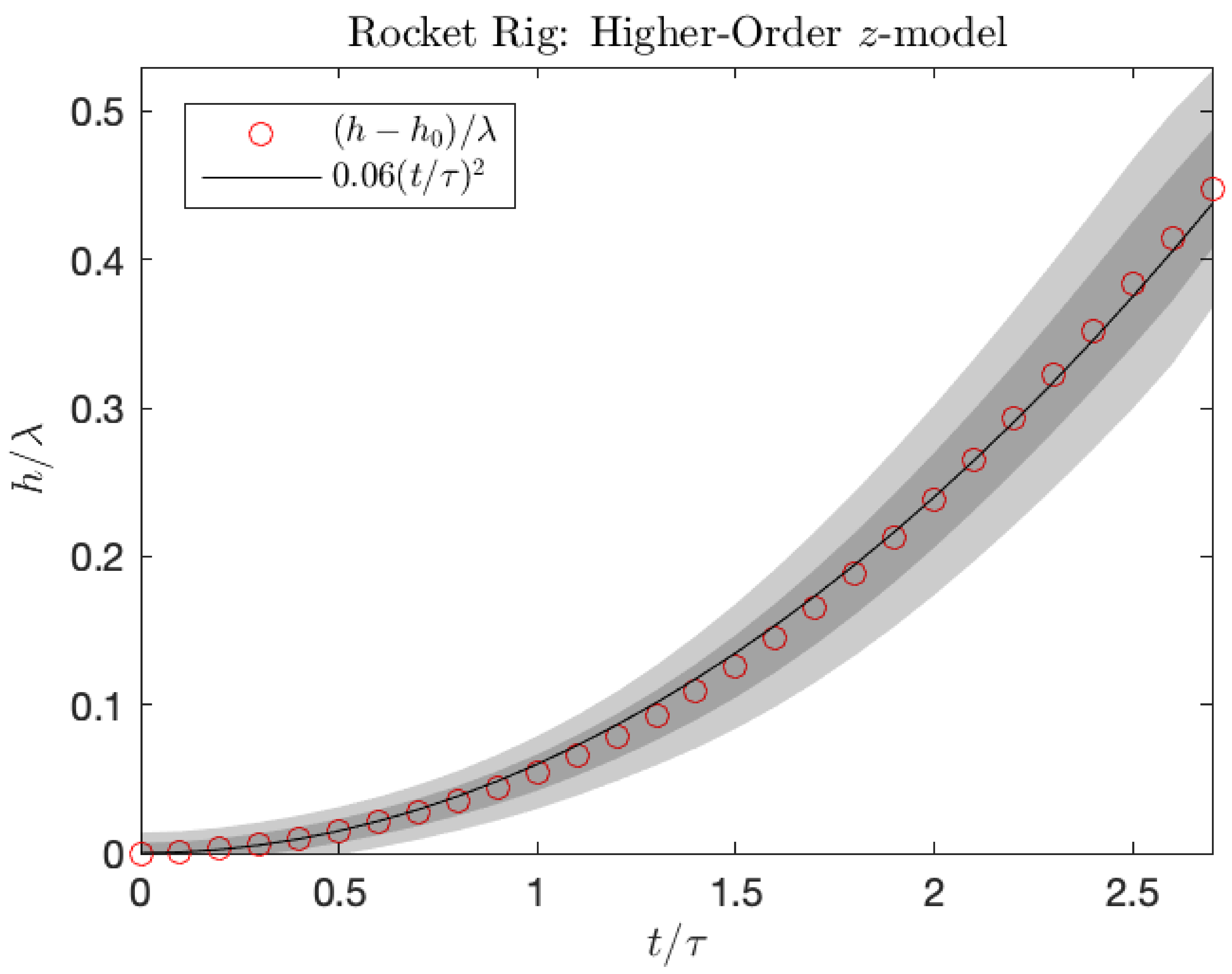
Rocket Rig

We begin with randomized initial data

$$z_3 = \mathrm{Re} \sum_{k_1,k_2=-N}^{N} c_{k_1,k_2} e^{2\pi i(k_1 s_1 + k_2 s_2)}$$

where the coefficients c_{k_1,k_2} are sampled from a normal distribution.

We averaged the result of 100 runs, using parameters that match Read's NaI/Pentane experiment.



Width of mixing layer, averaged over 100 runs. The mean is shown in red, and the grey bands show one and two standard deviations. Here λ is a characteristic length scale and $\tau = (\lambda/Ag)^{1/2}$ is the characteristic time scale. The simulations and the experiment were both run for 60ms, or approximately 2.7τ.

Comparison With Experiments

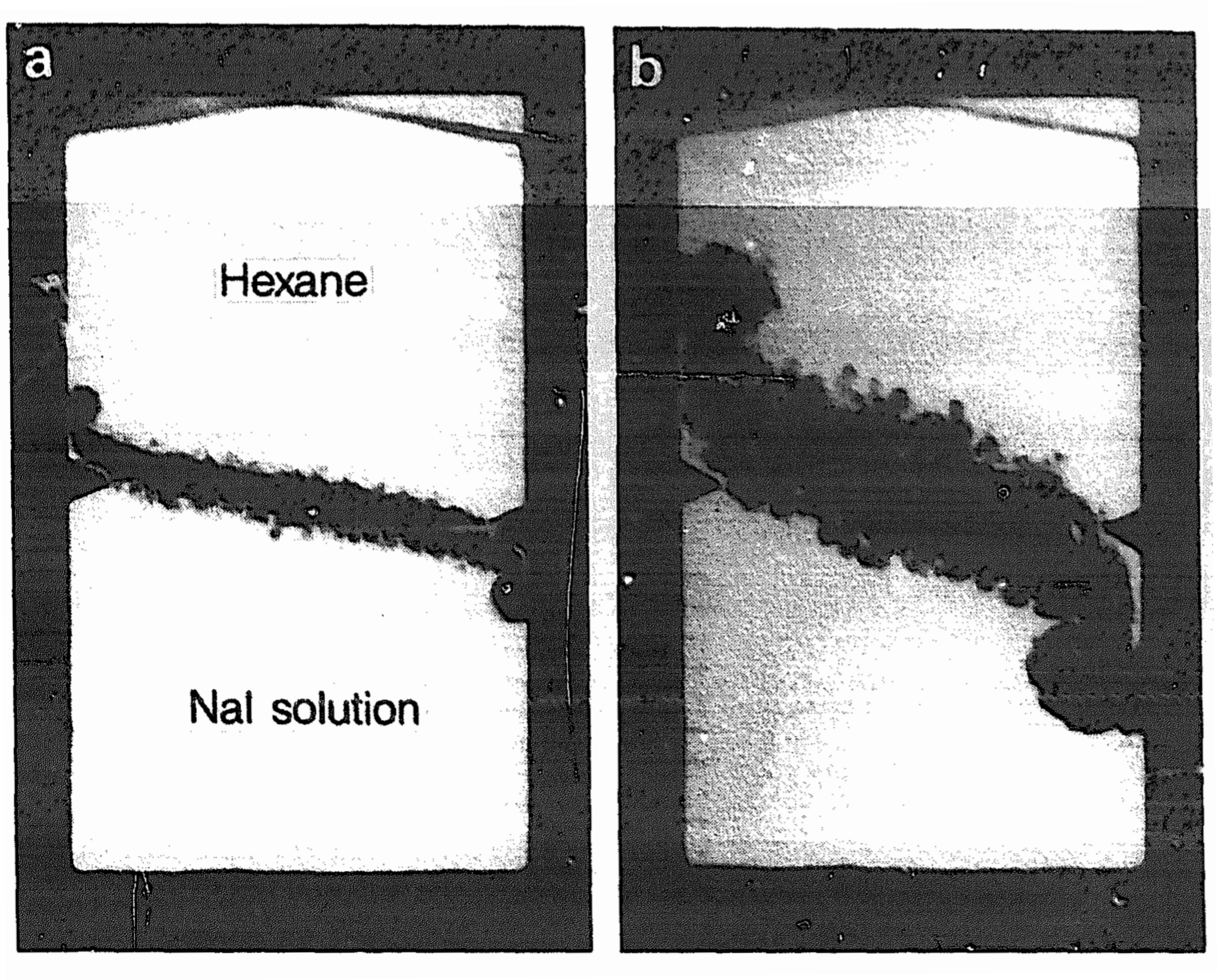
Tilted Rocket Rig

- This experiment is similar to the rocket rig, but the tank is tilted at a slope of 1/10.
- Our initial data has the form

$$z_3 = \frac{s_1}{10} + \operatorname{Re} \sum_{k_1,k_2=-N}^{N} c_{k_1,k_2} e^{2\pi i(k_1 s_1 + k_2 s_2)}$$

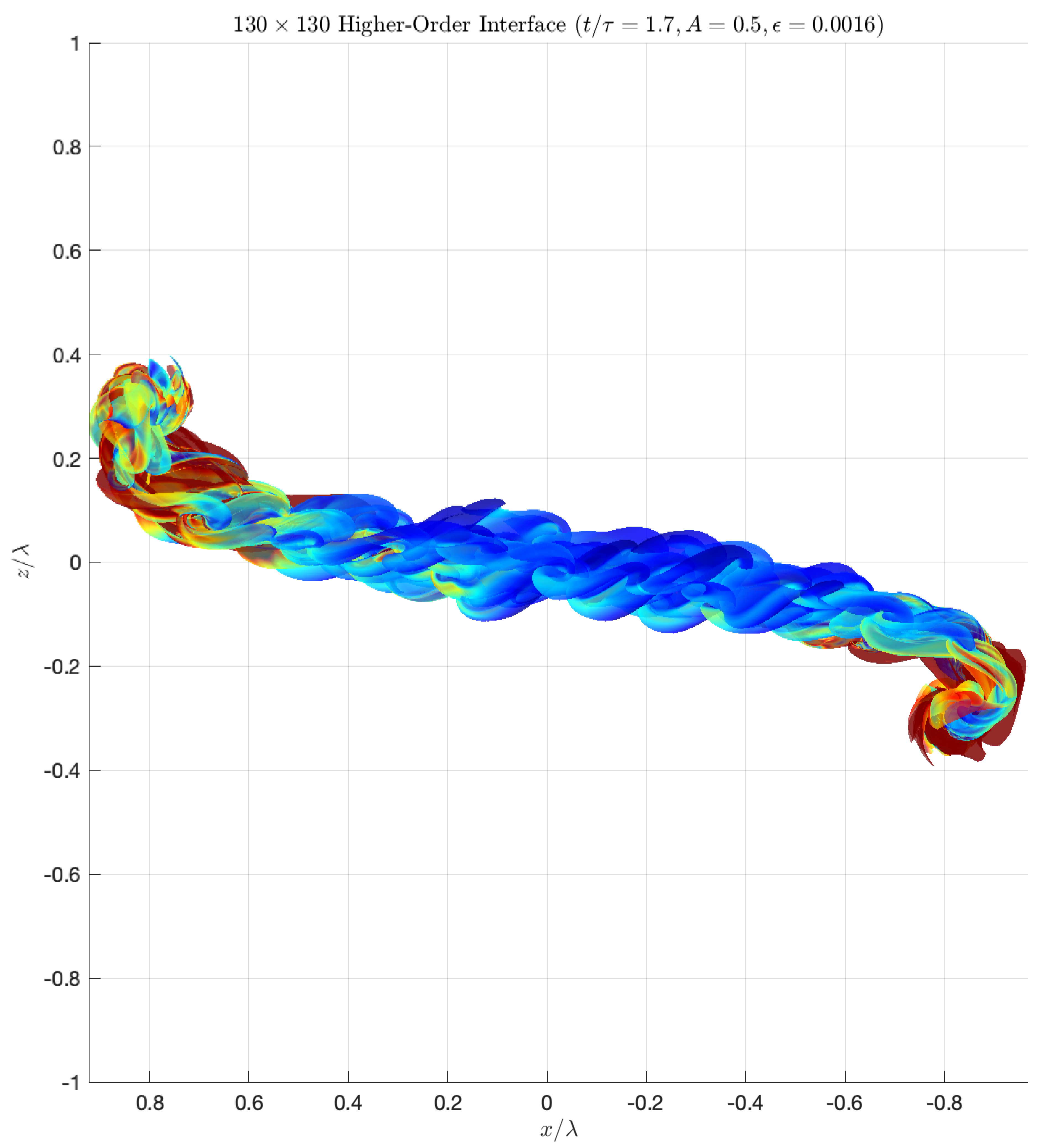
where the coefficients are again drawn from a normal distribution.

- The interface eventually flips over, so that the heavy fluid is on the bottom.

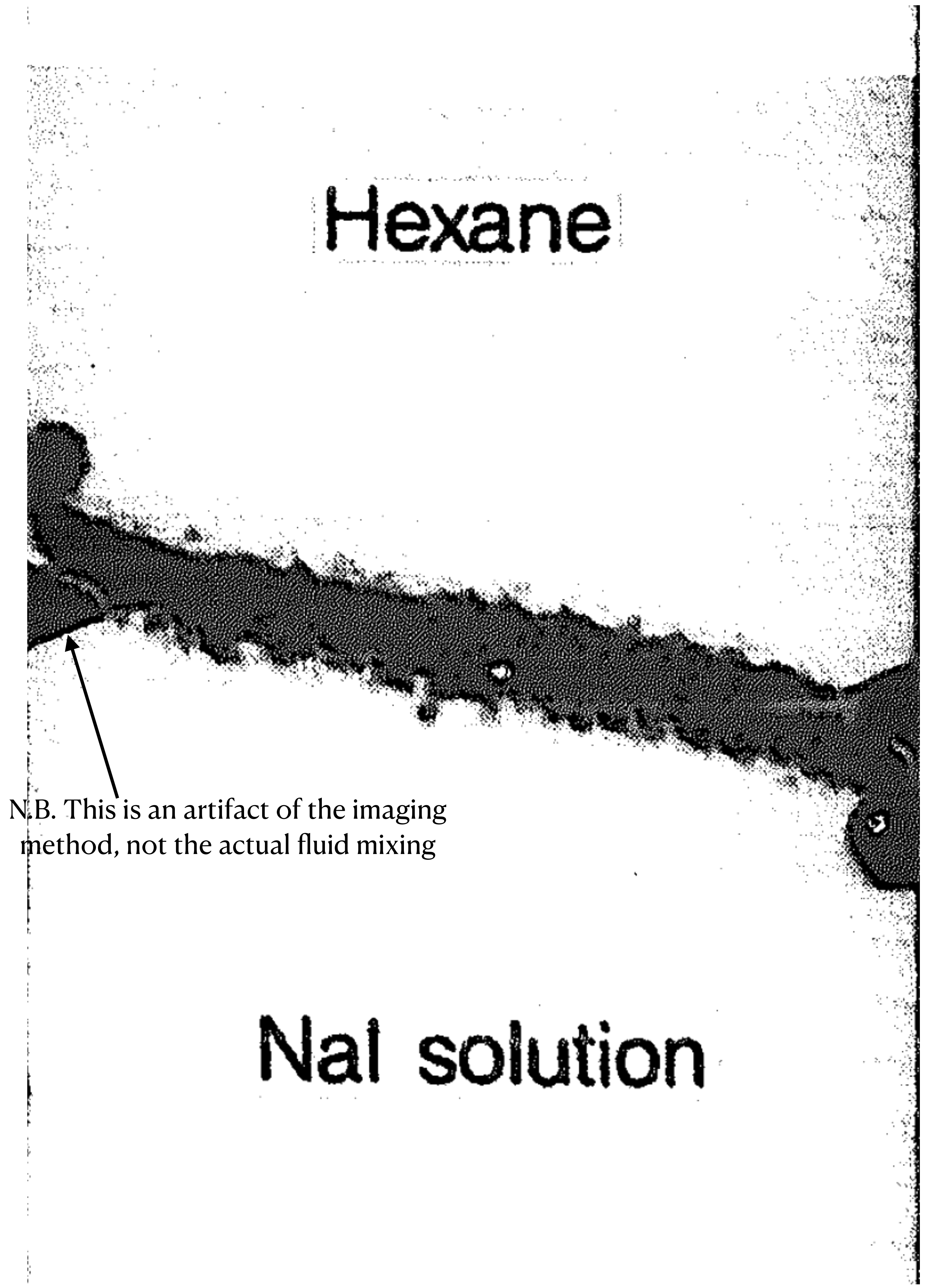


Experimental photographs by Youngs

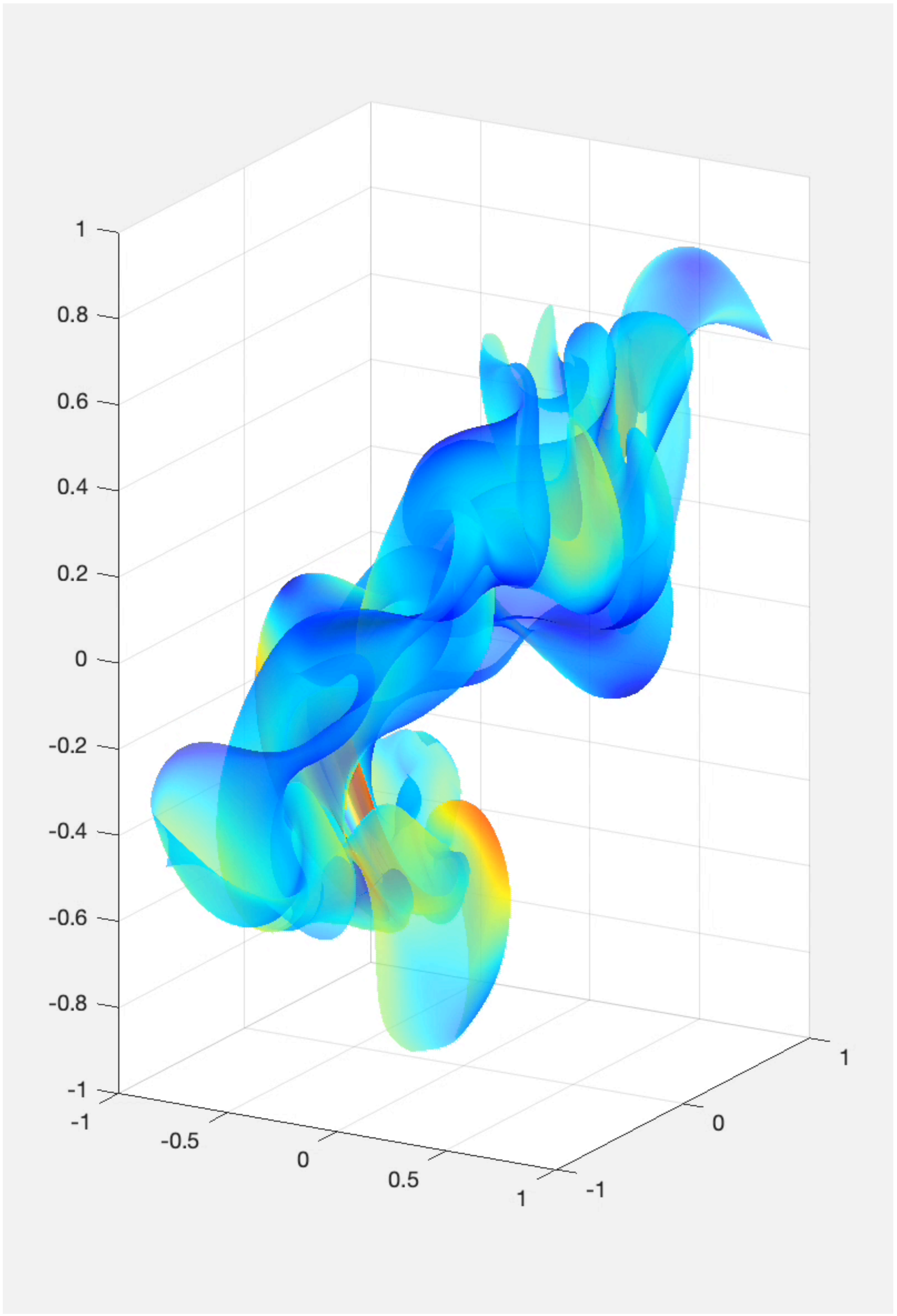
Comparison With Experiments



Side view of z-model at $t = 1.7\tau$



An experiment of Youngs, showing the tilted rig after 39ms, or approximately 1.7τ.

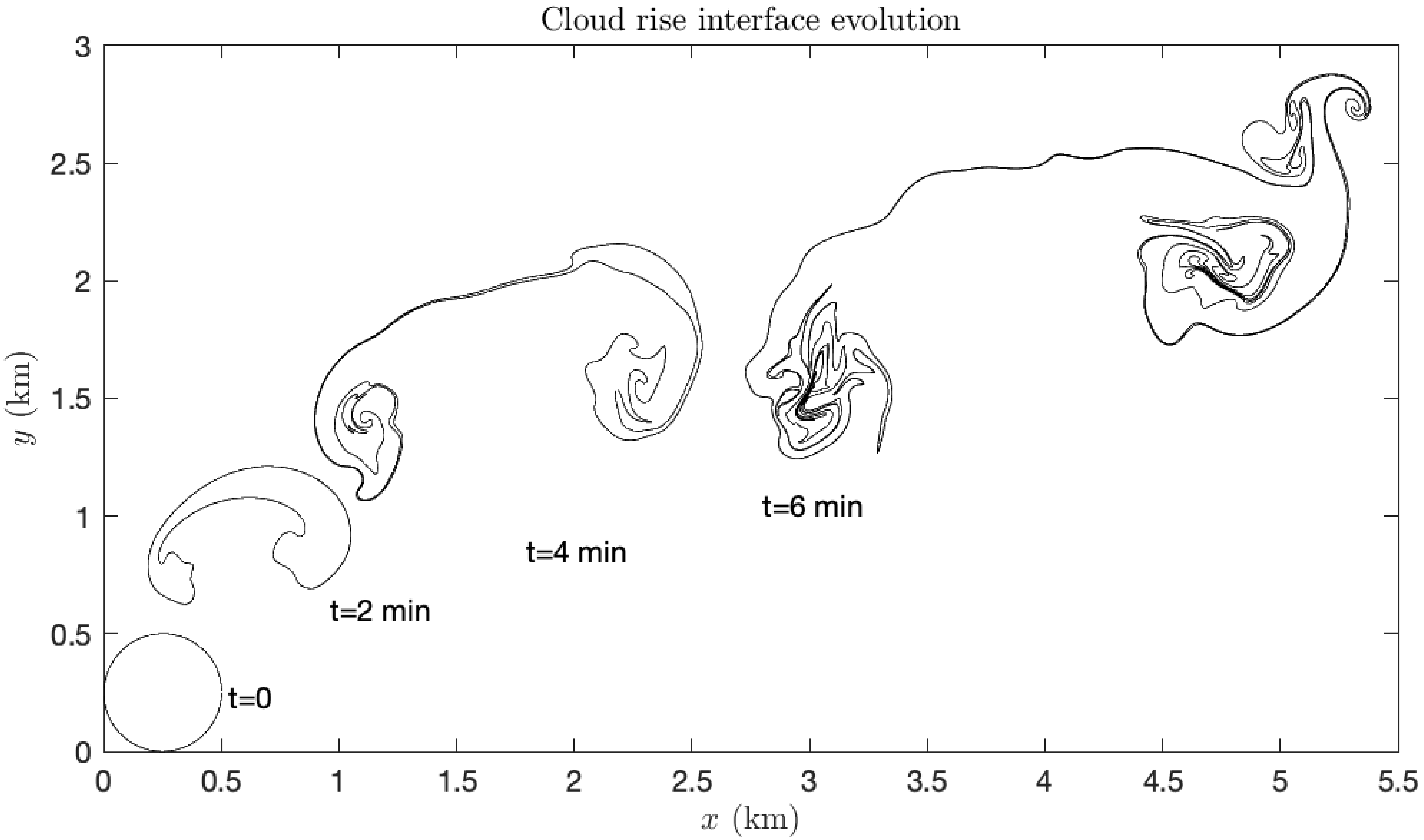


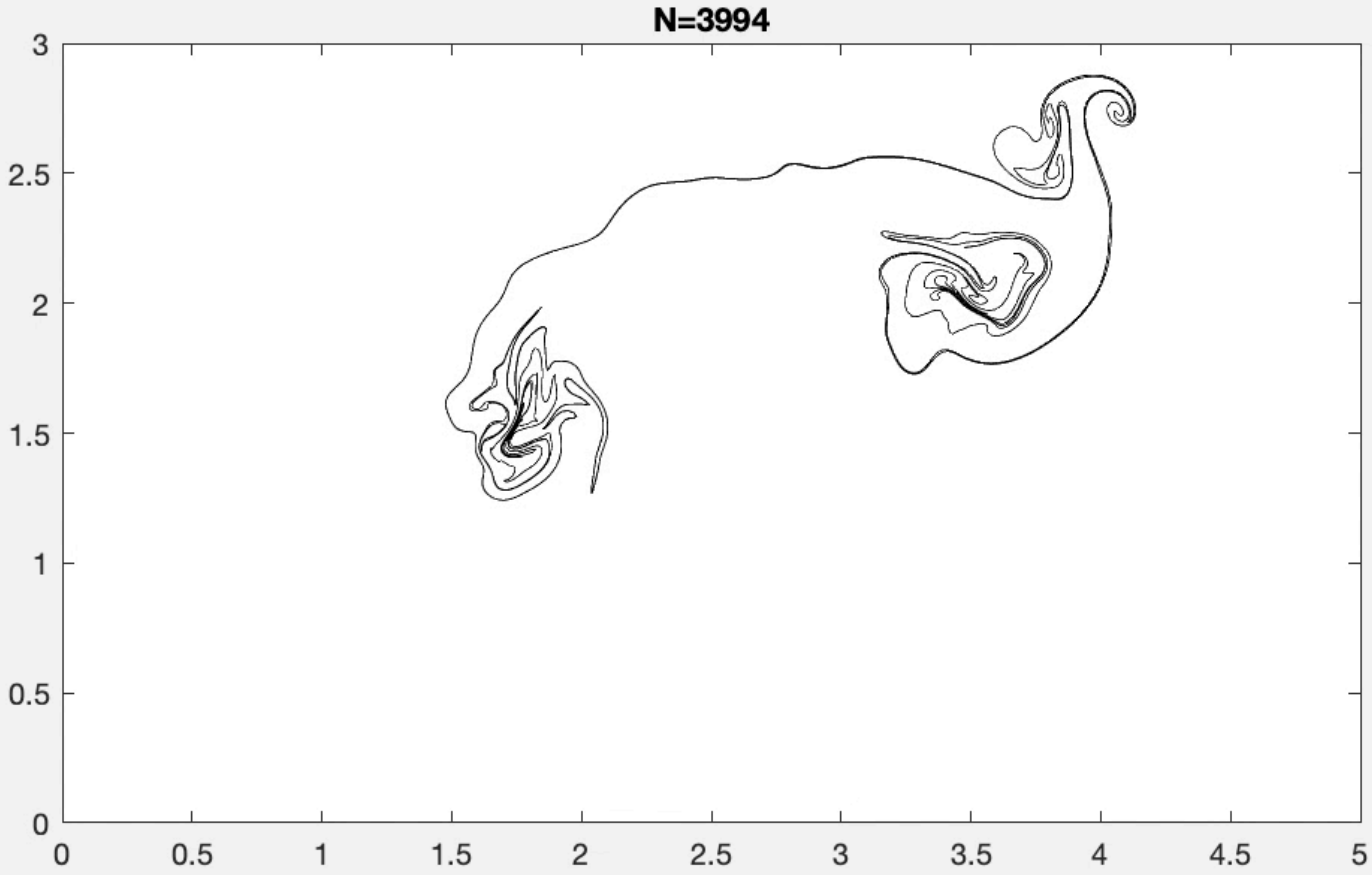
The tilted rig "flipping over" in our simulation.

Cloud Rise

z-model + density stratification + shear flow

- A sphere of warm air rising into density stratification w/ horizontal wind shear.
- Baroclinic generation of vorticity dominates the flow, so we can model this using the z-model

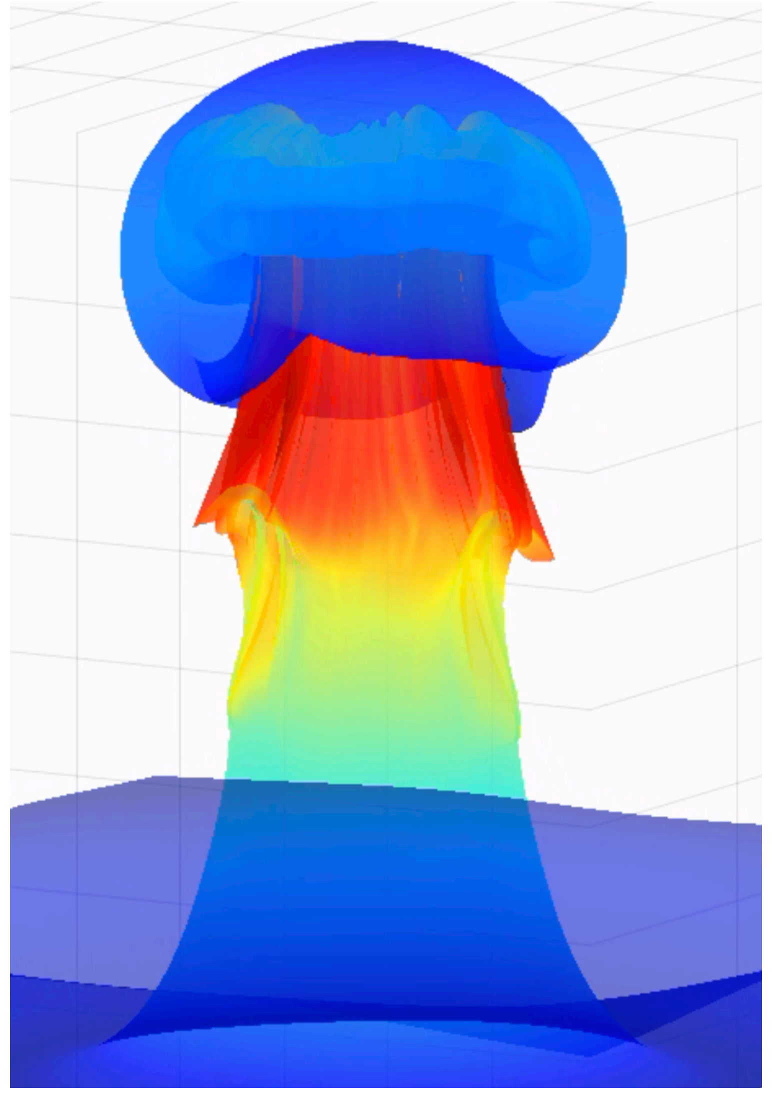



N=3994
3
2.5
2
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

Diffuse Interface

Sharp Interface

Rayleigh-Taylor instability in clouds vs. z-model



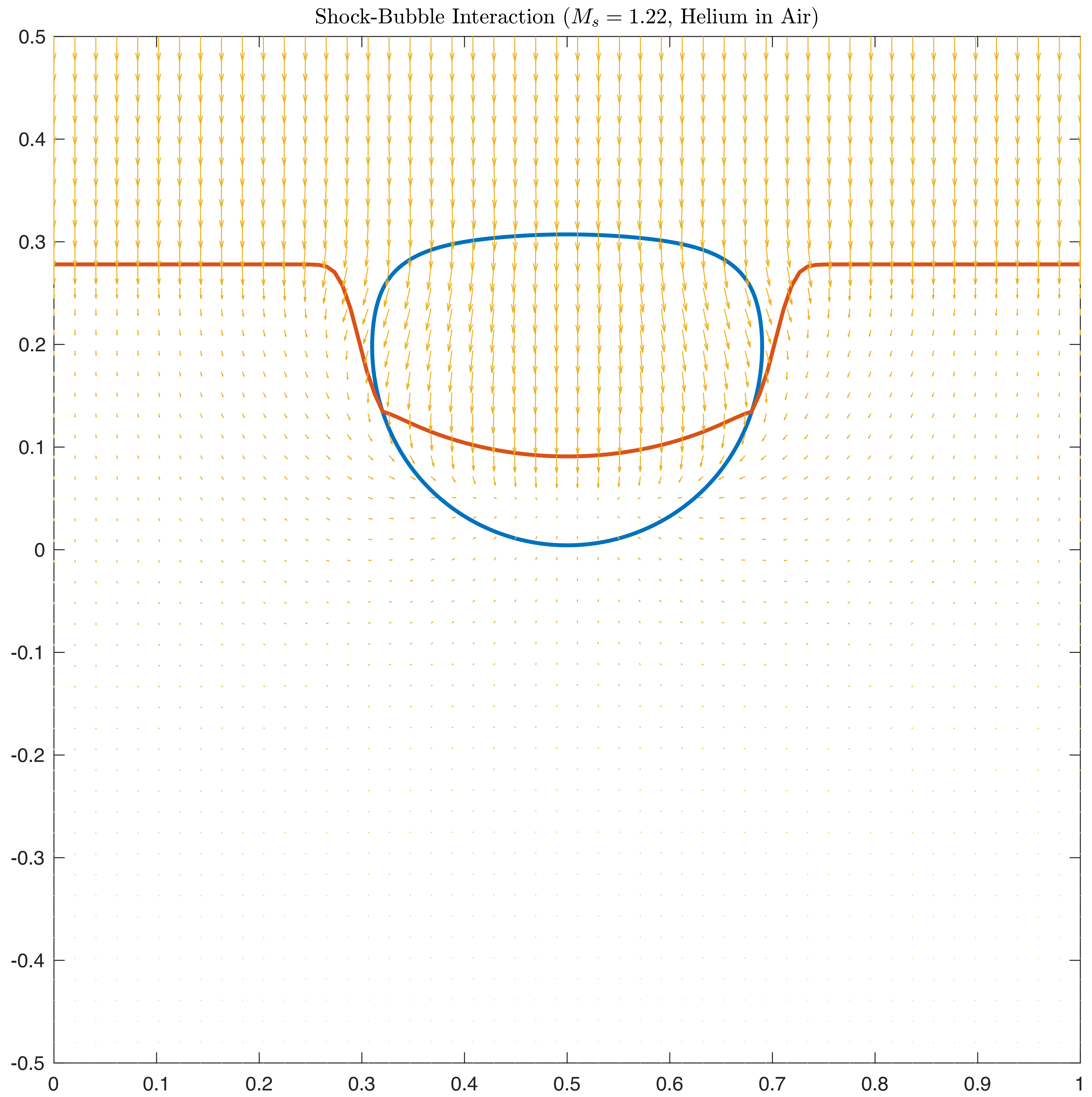
Rayleigh-Taylor Instability

1) water droplet impact
2) plume of superheated gas from atomic bomb
3) z-model simulation

Shock-Contact interaction

Z-model + GSD

- Coupling contact model to compressible flow field for RMI (Ramani and Shkoller '20)
- Coupling contact discontinuities to geometrical shock dynamics (GSD), with full velocity reconstruction.
- Comparison with experiments of Haas/Sturtevant on shock-bubble interaction



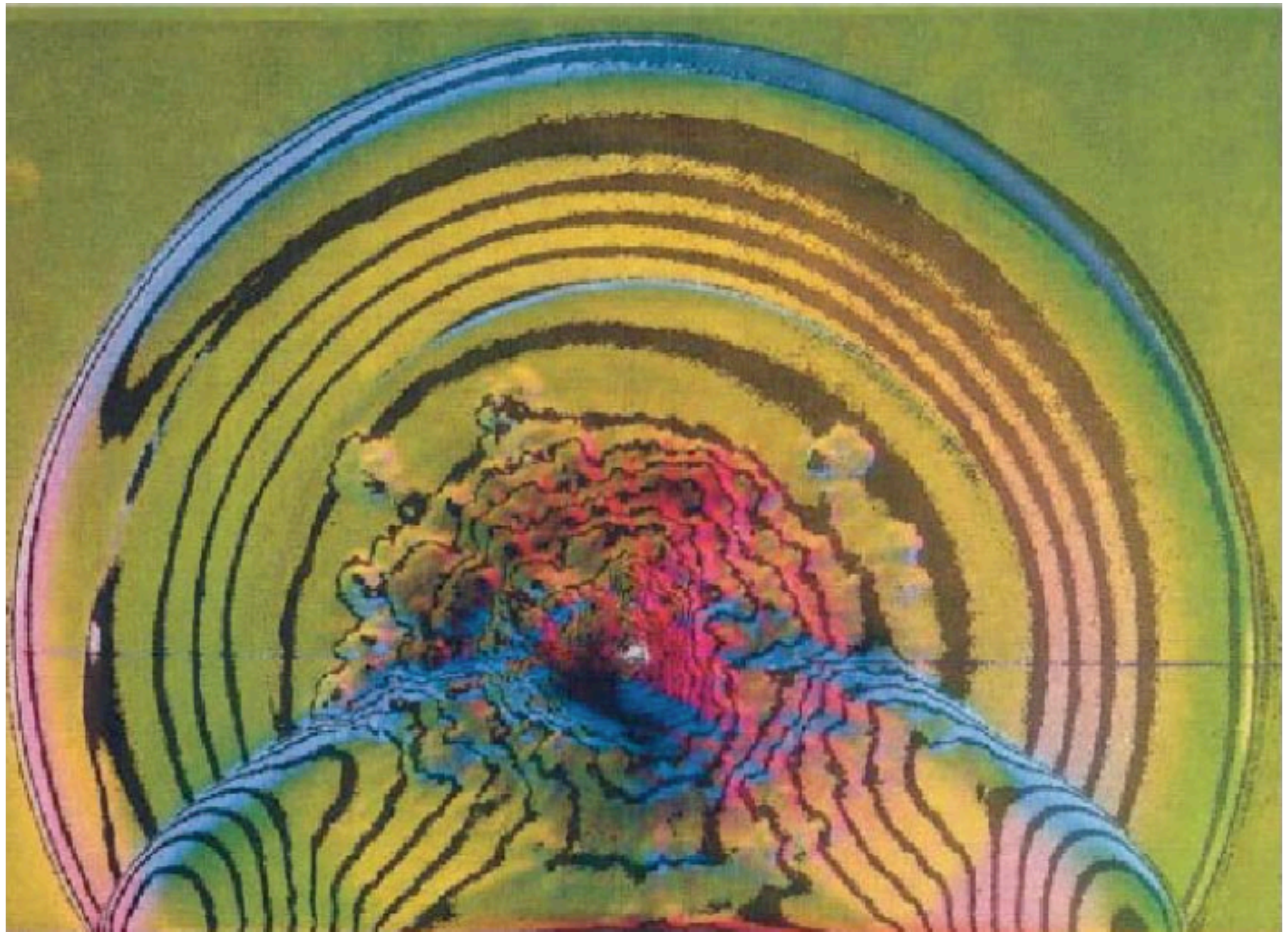
Shocks

Geometrical Shock Dynamics (Whitham, Schwendeman)

- Interface model for the shock position and shock Mach number

$$M = \frac{\text{shock speed}}{\text{sound speed}}.$$

- Models shock wave propagating into quiescent gas (e.g. blast wave from explosion, shock wave from supersonic airplane).



Schlieren/Interferometer image of blast wave (Kleine/Takayama)

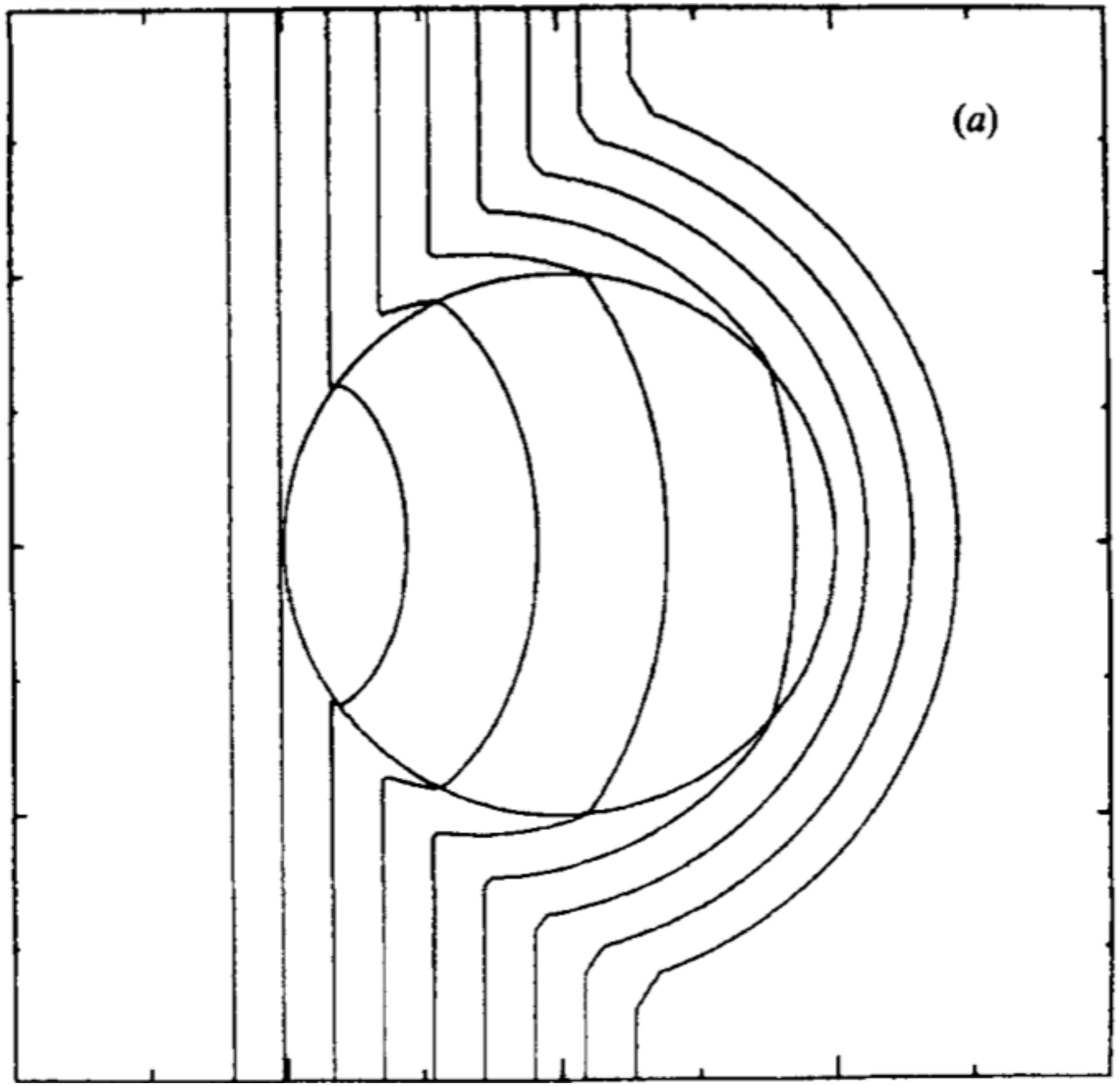
Shocks

Nonuniform Media

- When the sound speed changes, the Mach number changes too:

$$\Delta M = \frac{M\cos\theta}{\beta(M,\gamma)}\frac{\Delta c}{c}.$$

- In particular, where the shock encounters a density discontinuity, the shock develops a corner.



Shock propagating through sphere of lighter gas (Schwendeman)

Shocks and Contacts

Coupling the z-model and the kinematic shock model

- The contact is affected by the shock via baroclinic vorticity production:

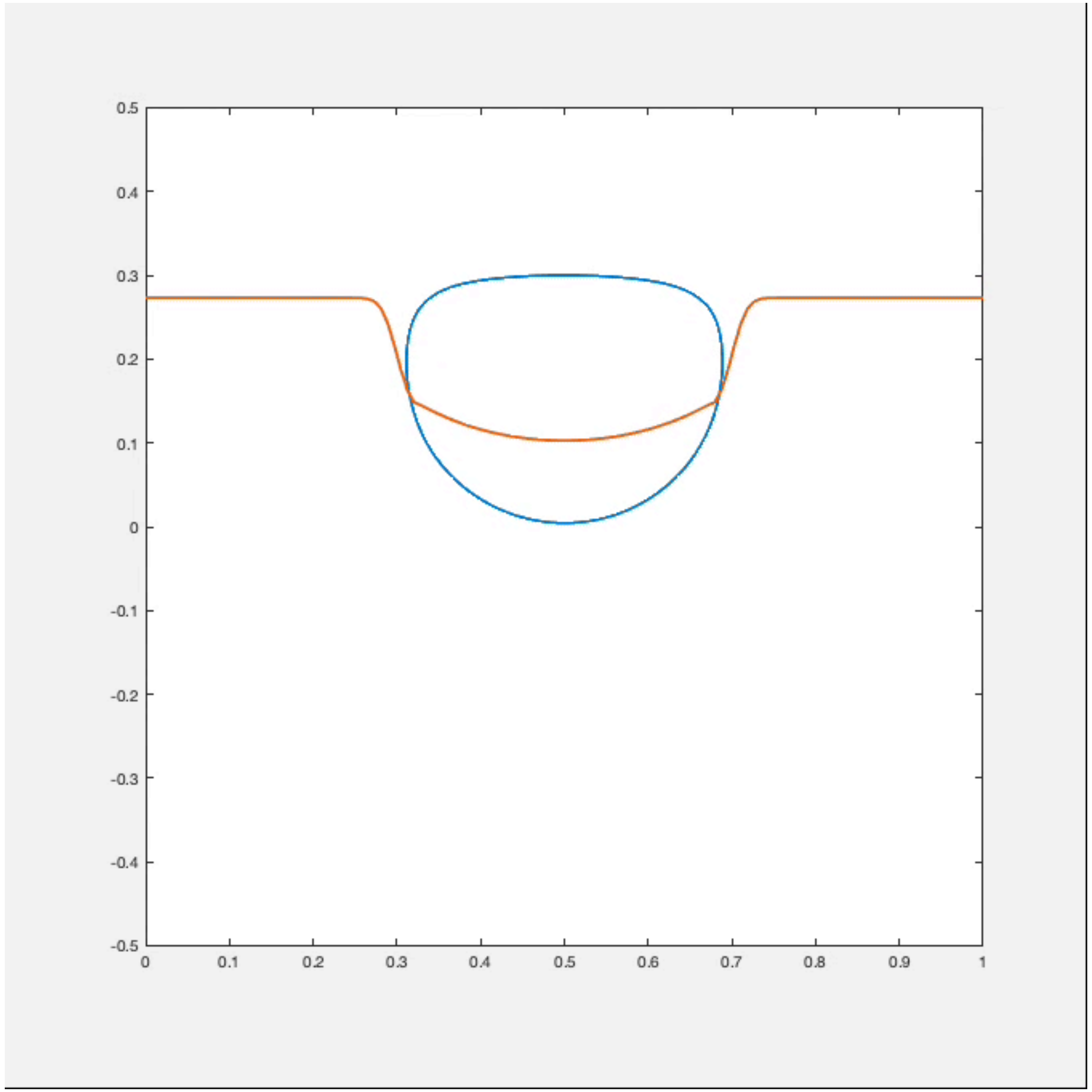
$$\Delta\omega = \left(\frac{1}{\rho^+} - \frac{1}{\rho^-}\right)\Delta p \sin\theta.$$

- The shock is affected by the contact via the change in sound speed:

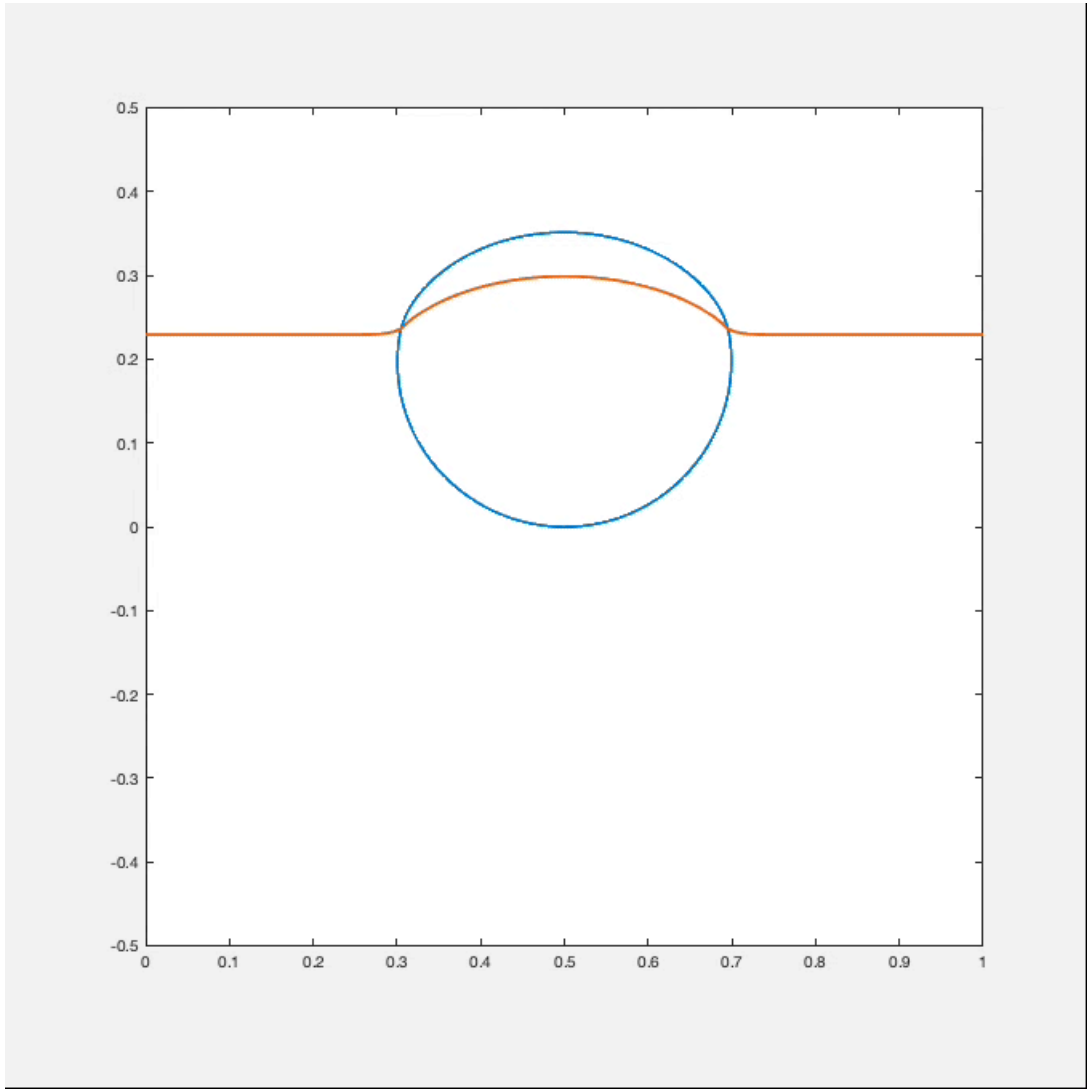
$$\Delta M = \frac{M\cos\theta}{\beta(M,\gamma)}\frac{\Delta c}{c} \qquad\qquad c^2 = \frac{\gamma p}{\rho}.$$

- The full velocity is reconstructed from the amplitudes of vorticity and compression, using our boundary integral formulation.

Mach 1.22 shock propagating through helium bubble in air;
our simulation vs. experiments of Haas/Sturtevant.



Air-Helium

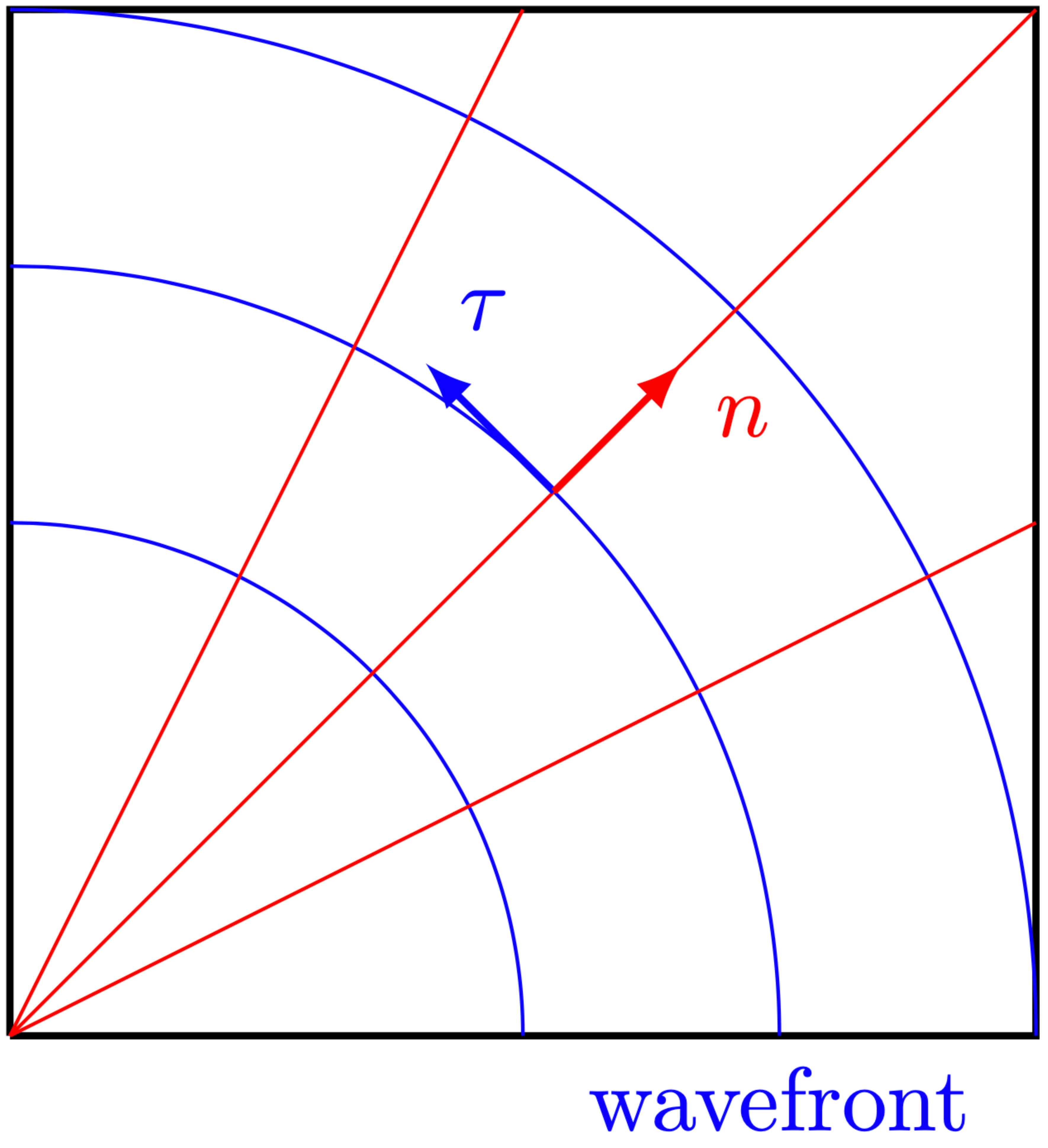


Air-Freon

Nonlinear Sound Waves

The wza-model

- Compressible Euler equations have three distinct characteristic velocities.
- Introduce new geometric coordinate system that moves with fast acoustic characteristic
- Simultaneously solve for both physical variables and geometric unknowns



Characteristic Surfaces

Fast Acoustic Characteristic

- Geometric coordinates

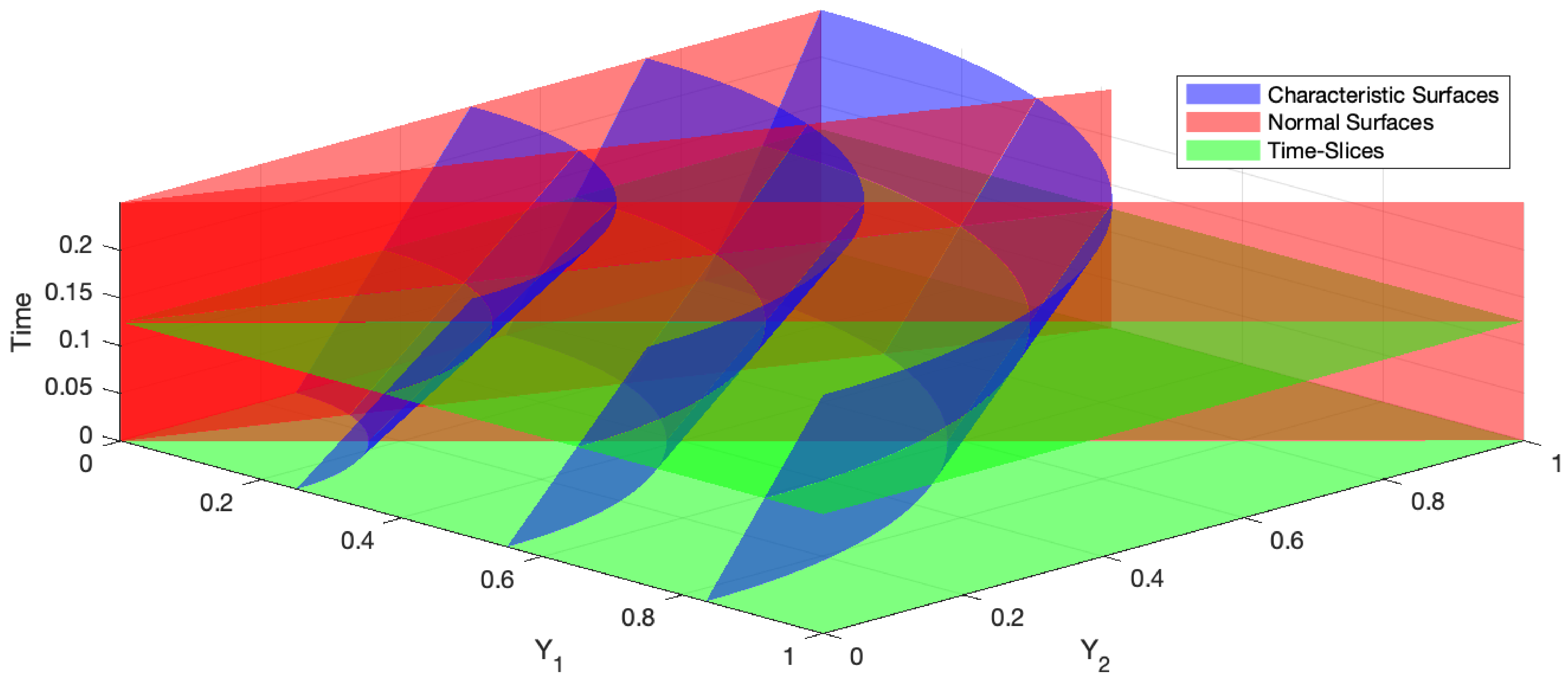
$$y_1 = \eta_1(x_1, x_2, t), \qquad y_2 = \eta_2(x_1, x_2, t)$$

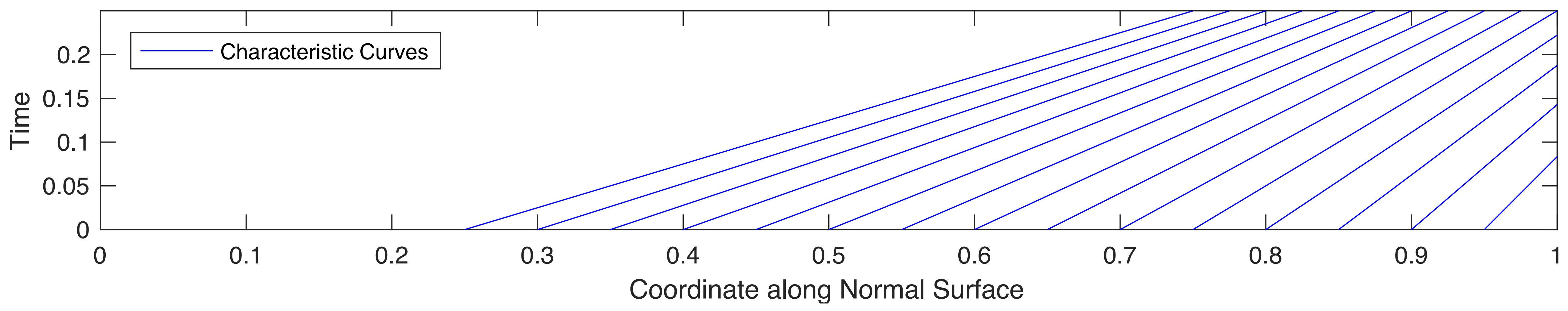
- Wavefronts given by $x_1 = \text{const.}$
- Tangent and normal directions

$$\tau = \frac{(\partial_2\eta_1, \partial_2\eta_2)}{|(\partial_2\eta_1, \partial_2\eta_2)|}, \ n = \frac{(\partial_2\eta_2, -\partial_2\eta_1)}{|(\partial_2\eta_1, \partial_2\eta_2)|}$$

- Time-evolution given by

$$\partial_t \eta = u \circ \eta + (c \circ \eta) n$$





Geometric Riemann Variables

The wza-model

- Riemann invariants given by $w = u \cdot n + c/\alpha$, $z = u \cdot n - c/\alpha$, $a = u \cdot \tau$.
- In 1d, w is constant along fast acoustic characteristic; in 2d, the evolution of w only depends on derivatives tangent to wavefront.

Evolution of wavefront

Evolution of geometric Riemann variables along wavefront

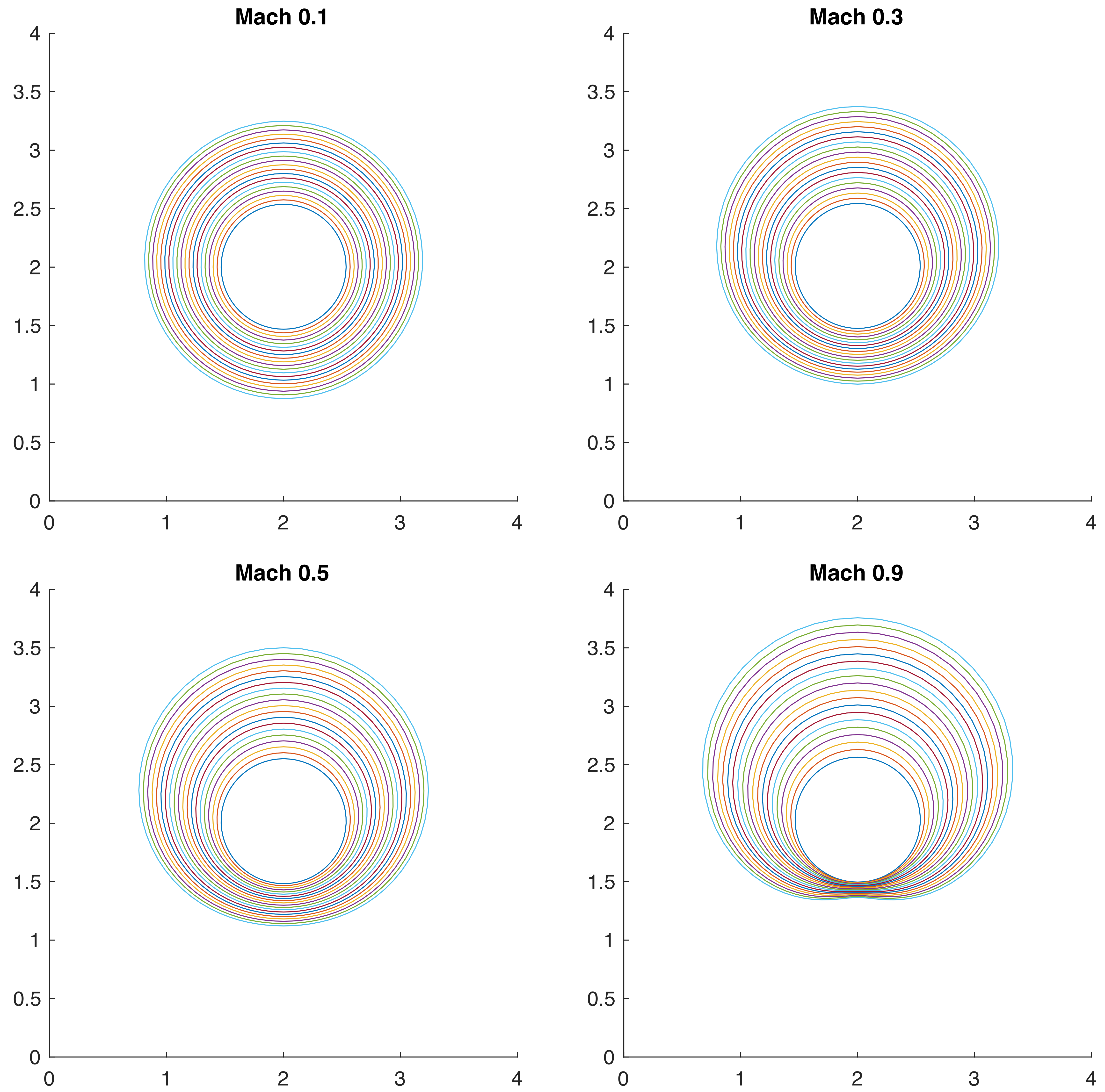
Normal derivatives of subdominant Riemann variables are negligible

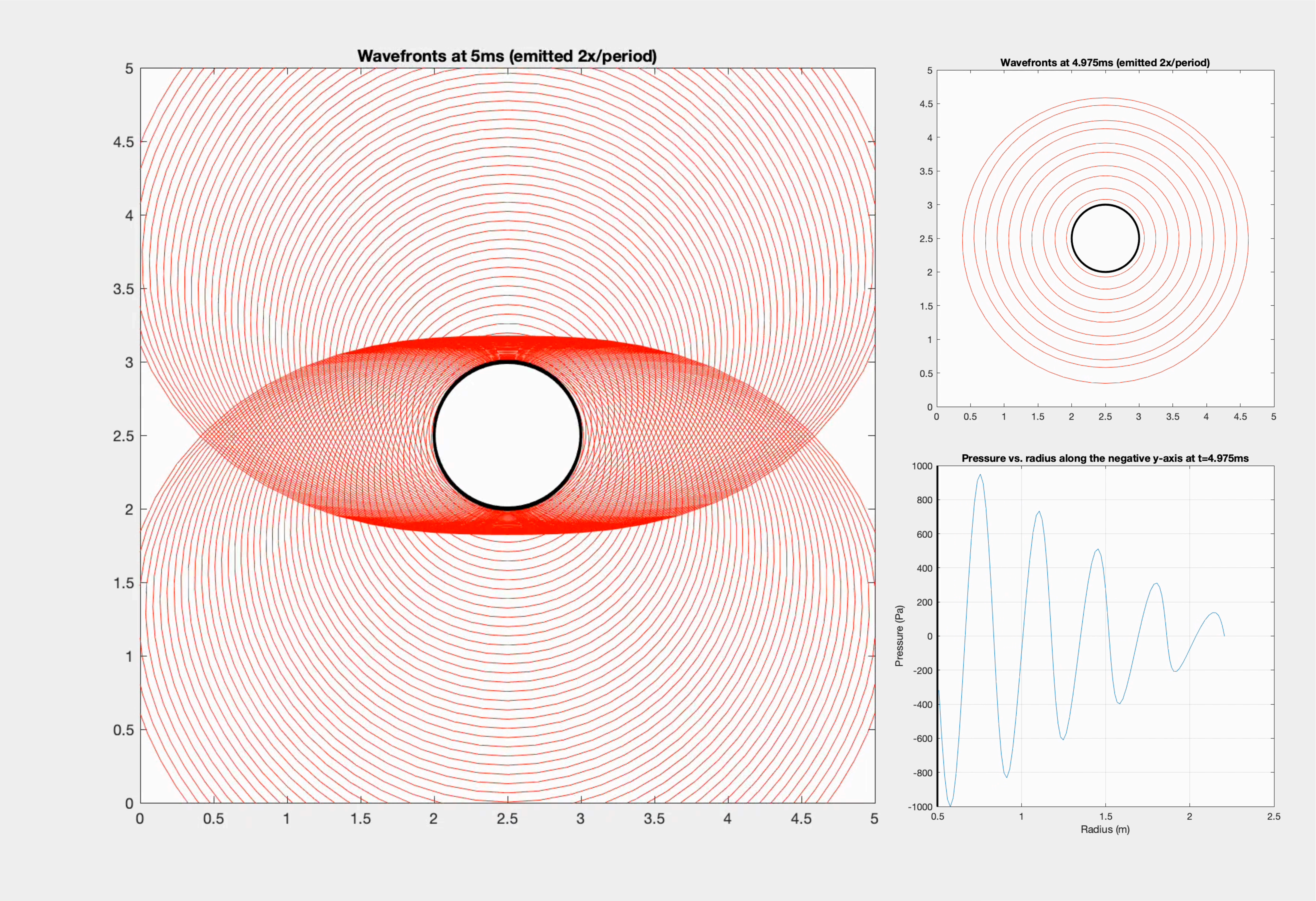
$$
\begin{cases}
\partial_t \eta = a\tau + \left(\frac{1+\alpha}{2} w + \frac{1-\alpha}{2} z\right) n \\
\partial_t w = -a\partial_\tau\left(\frac{1+\alpha}{2} w + \frac{1-\alpha}{2} z\right) - \frac{\alpha}{2}(w-z)\partial_\tau a + \left(\frac{\alpha}{4}(w^2 - z^2) - a^2\right)\kappa + a\partial_n n \cdot \tau \\
\partial_t z = -a\partial_\tau\left(\frac{1+\alpha}{2} w + \frac{1-\alpha}{2} z\right) + \frac{\alpha}{2}(w-z)\partial_\tau a - \left(\frac{\alpha}{4}(w^2 - z^2) + a^2\right)\kappa + \alpha(w-z)\partial_n z - a\partial_n n \cdot \tau \\
\partial_t a = -\frac{\alpha}{4}(w-z)\partial_\tau(w-z) + \frac{1}{2}(w+z)\left(\frac{1+\alpha}{2} w + \frac{1-\alpha}{2} z\right) + \frac{1}{2}a(w+z)\kappa + \frac{\alpha}{2}(w-z)\partial_n a
\end{cases}
$$

Sound Waves

Interface Model

- Following the fast characteristic, we obtain an interface model for acoustic waves.
  - Approaches geometrical optics in the low Mach number limit
  - Tracks Riemann invariants along fast acoustic characteristics
- Captures nonlinear effects (steepening of acoustic waves into shock waves)





Continuous "emission" of wavefronts allows for reconstruction of sound wave profiles

Future Work

- Models in development:
  - Interface model for compressible vorticity waves and contact discontinuities.
  - Interface model for shock waves using our framework.
  - Steepening of nonlinear sound waves into shock waves.
  - Incompressible contact model including viscous effects
- Coupling of interface models:
  - Shock-contact interaction
  - Shock-sound wave interaction


Thank You!