Multiscale Vortex Methods

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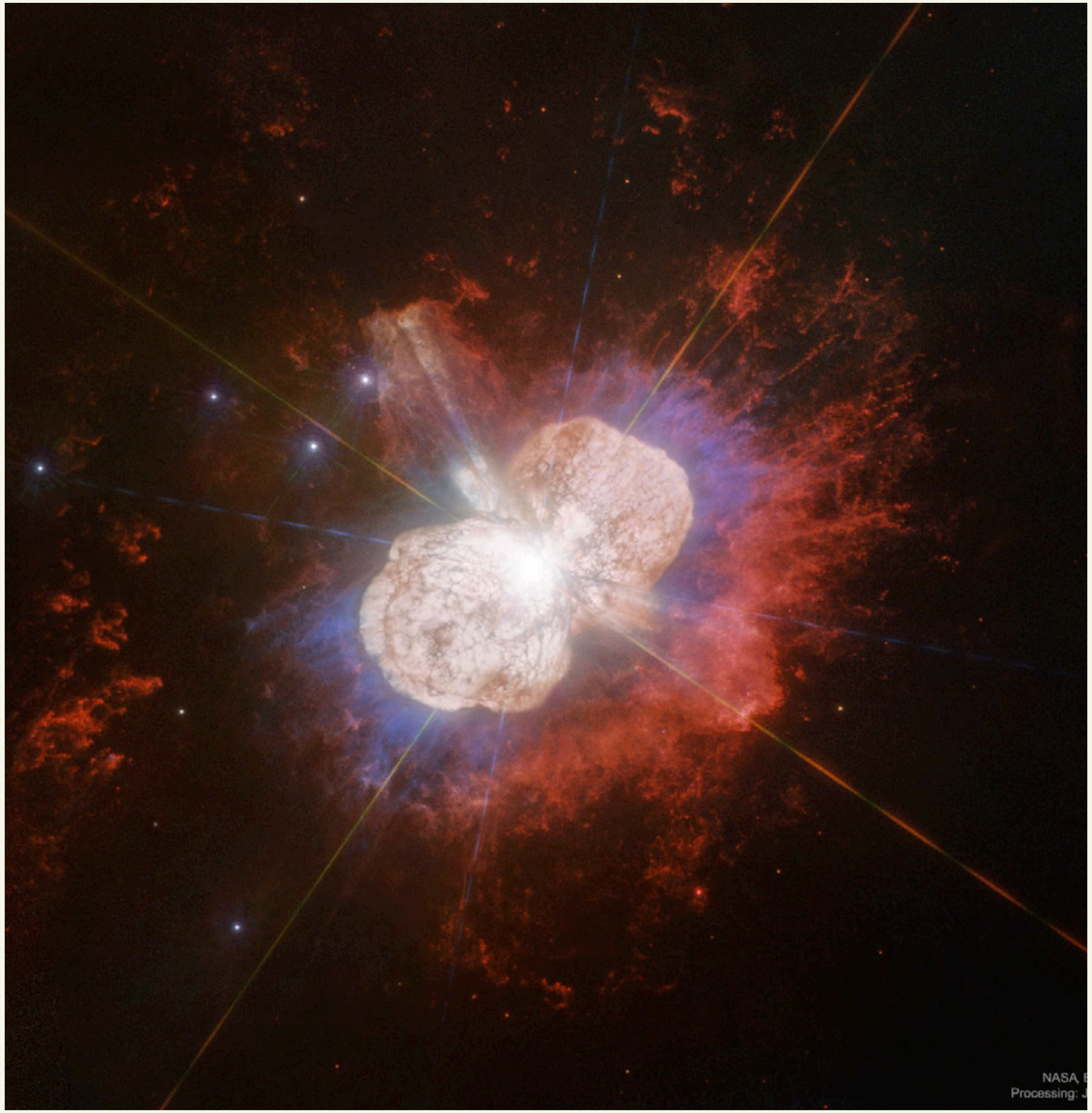
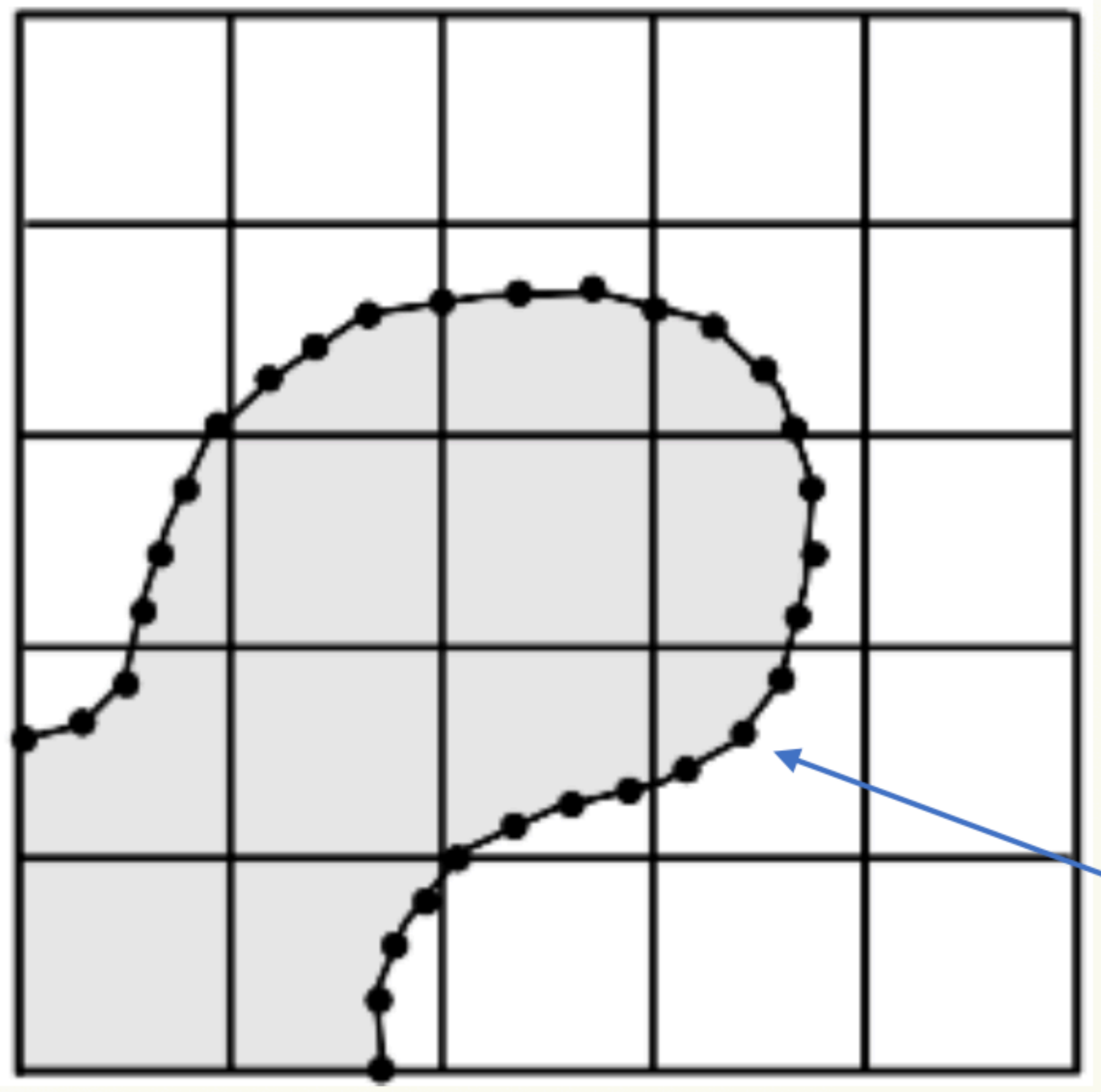
Methodology

Vortical flows via interface models

- Fluid systems often dominated by surfaces of discontinuity—shocks, contacts, vortex sheets.
- Interface methods offer geometric flexibility, computational efficiency, and concise representation.
- Less useful at scales where flow is dominated by mixing or turbulence.



Front Tracking vs. Interface Models

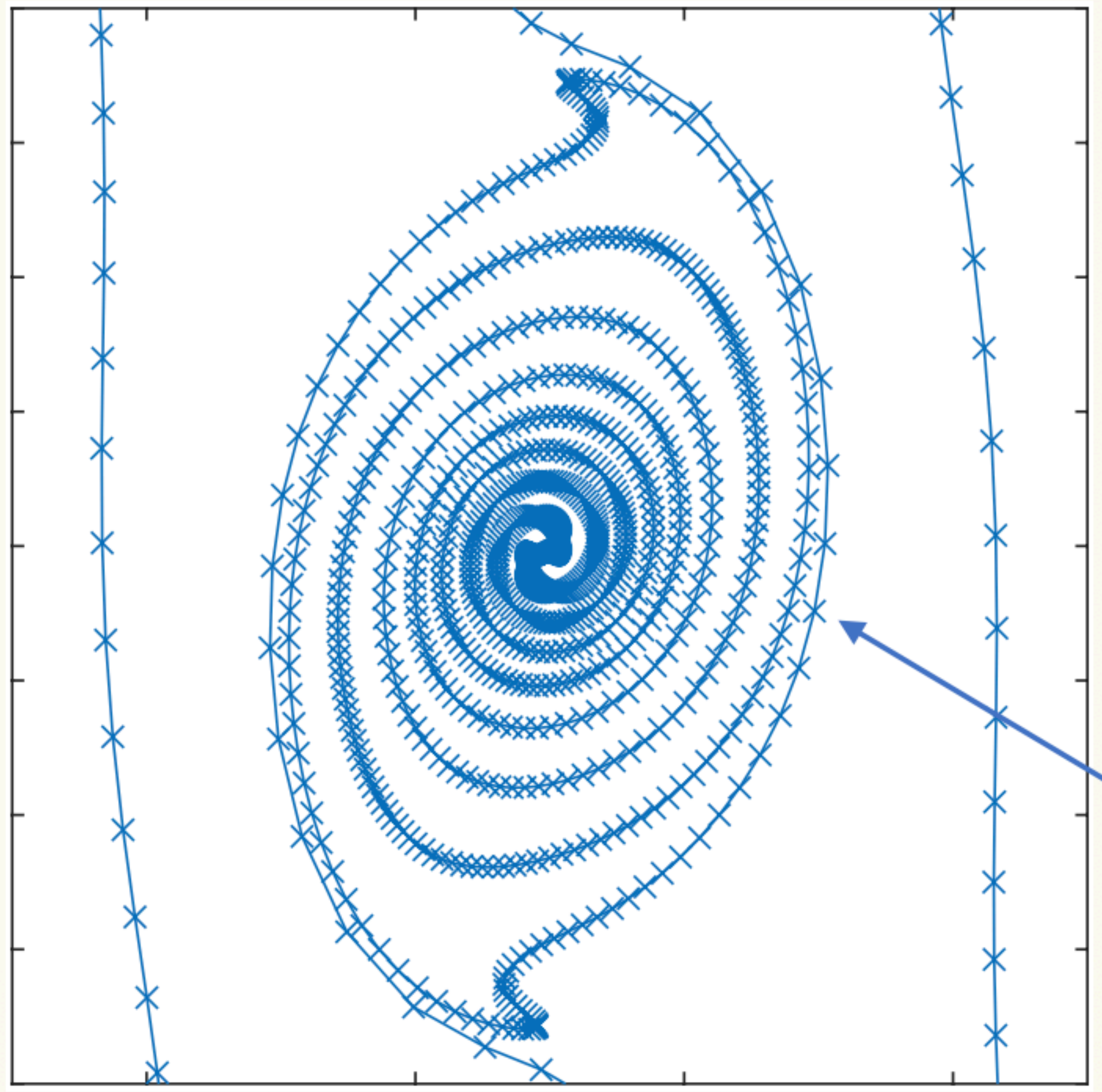


Flow variables:

$$\begin{cases} \rho(x, y, t) \\ u(x, y, t), v(x, y, t) \\ p(x, y, t) \end{cases}$$

Marker points advect passively

Schematic of front-tracking method



Interface variables:

$$\begin{cases} x(s, t), y(s, t) \\ \omega(s, t) \\ \vdots \end{cases}$$

Marker points are active variables

Schematic of interface model

Eulerian simulation + front tracking

- Tracks density and velocity on 2D mesh + advects 1D interface with fluid velocity
- Accuracy depends on resolution of both 2D Eulerian mesh and 1D interface mesh

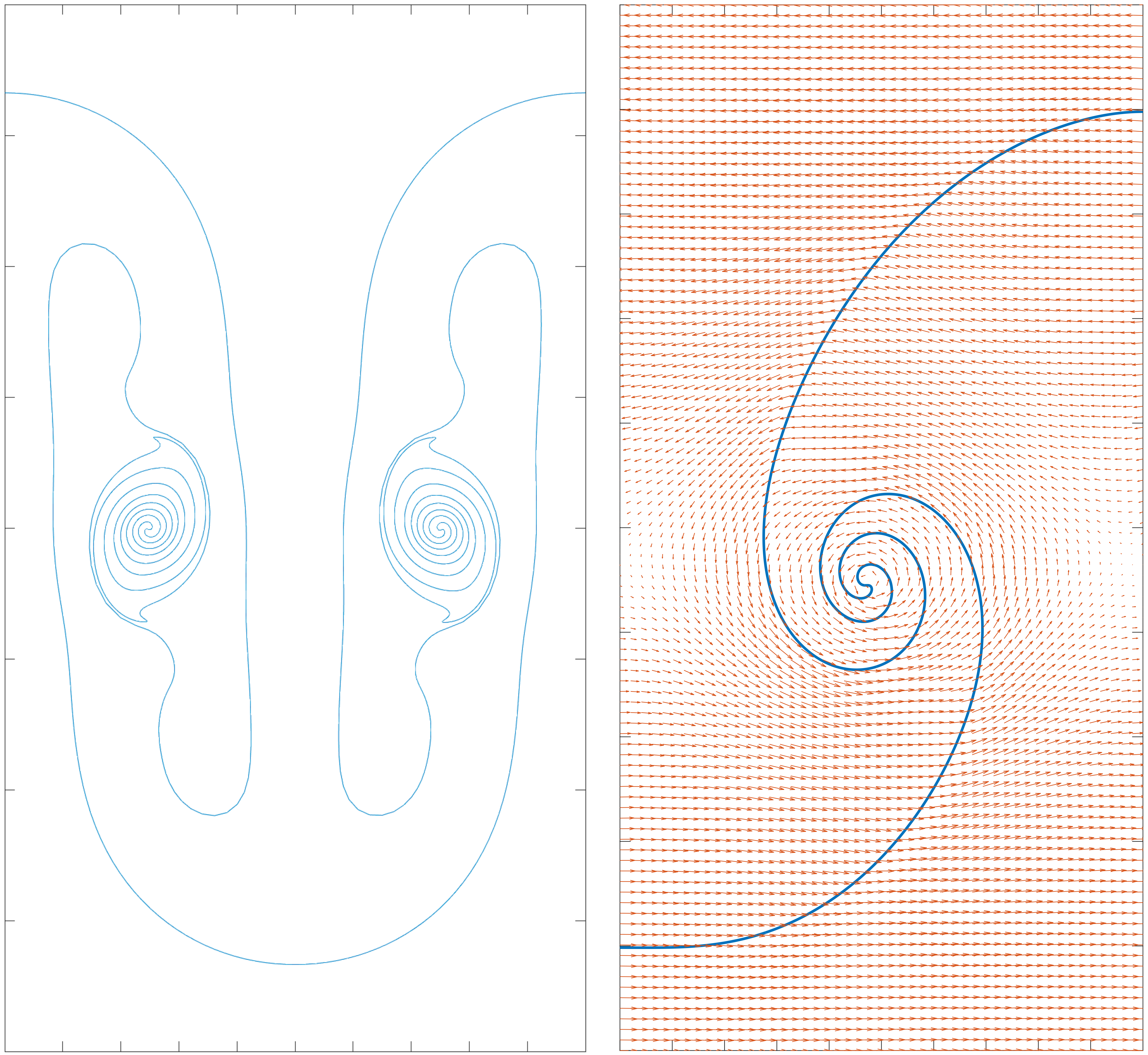
Interface model simulation

- Self-contained equations for interface location and quantities defined on the interface

The z-Model

(P. and Shkoller, JFM '23)

- Interface model for the interface position and amplitude of vorticity.
- Models the deposition of vorticity on density discontinuities.
- Velocity reconstructed using nonlocal Biot-Savart integral, *assuming incompressible and irrotational flow in the bulk.*



z-model interface, Rayleigh-Taylor instability with reconstructed velocity

Equations of Motion

Two-phase compressible flow

- Two regions $\Omega^+(t), \Omega^-(t)$ separated by a free surface $\Gamma(t)$.
- Compressible Euler:

$$
\begin{aligned}
\partial_t \rho^\pm + \nabla \cdot (\rho^\pm \boldsymbol{u}^\pm) &= 0, && \boldsymbol{x} \in \Omega^\pm(t) \\
\partial_t (\rho^\pm \boldsymbol{u}^\pm) + \nabla \cdot (\rho^\pm \boldsymbol{u}^\pm \otimes \boldsymbol{u}^\pm) + \nabla p^\pm &= -\rho^\pm g \boldsymbol{e}_y, && \boldsymbol{x} \in \Omega^\pm(t) \\
\partial_t (\rho^\pm e^\pm) + \nabla \cdot ((\rho^\pm e^\pm + p^\pm) \boldsymbol{u}^\pm) &= -\rho^\pm g v^\pm, && \boldsymbol{x} \in \Omega^\pm(t)
\end{aligned}
$$

- Jump conditions:

$$
[\![\boldsymbol{u} \cdot \boldsymbol{n}]\!] = 0, \qquad [\![p]\!] = 0,
$$

Equations of Motion

Momentum equation decomposition

- Recall the definition of the distributional curl of a vector field $\mathbf{v} \in L^1_{loc}(\mathbb{R}^2)$,

$$\langle \nabla \times \mathbf{v}, f \rangle := -\int_{\mathbb{R}^2} \mathbf{v} \cdot \nabla^{\perp} f, \qquad \forall f \in C_c^{\infty}(\mathbb{R}^2)$$

- Taking the distributional curl of the two-phase momentum equation gives

$$0 = \int_{\Gamma} [\![\rho(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}) + \nabla p]\!] \cdot \boldsymbol{\tau} f - \int_{\mathbb{R}^2} \rho\Big(\partial_t \boldsymbol{\omega} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} + \boldsymbol{\omega} \operatorname{div} \boldsymbol{u} + \tfrac{\nabla \rho \times \nabla p}{\rho^2}\Big) f$$

- This yields the ordinary vorticity equation in the bulk, plus a jump condition along the free surface $\Gamma(t)$.

Equations of Motion

Interface equations

- Dependent variables: interface parametrization $z(s,t)$, and tangential velocity jump $\mu(s,t) := [u](z(s,t),t)\cdot\partial_s z(s,t)$
- We can reduce the 'singular part' of the momentum equation results to the z-model!

$$\partial_t \mu = \left(\frac{\rho^+ - \rho^-}{\rho^+ + \rho^-}\circ z\right)\partial_s\left(|u\circ z|^2 - \frac{\mu^2}{4|\partial_s z|^2} - 2gz_2\right)$$

- Using a different method, Ramani and Shkoller '20 obtained a similar equation but assuming *constant densities* ρ^+, ρ^-.

Equations of Motion

Velocity decomposition

- Biot-Savart velocity:

$$\bar{u}(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\left(x - z(s,t)\right)^{\perp}}{|x - z(s,t)|^2}\mu(s,t)ds$$

- Background compressible velocity:

$$\tilde{u}(x,t) = u(x,t) - \bar{u}(x,t)$$

- Discontinuity in tangential velocity is *cancelled out*—$\tilde{u}(x,t)$ is continuous.

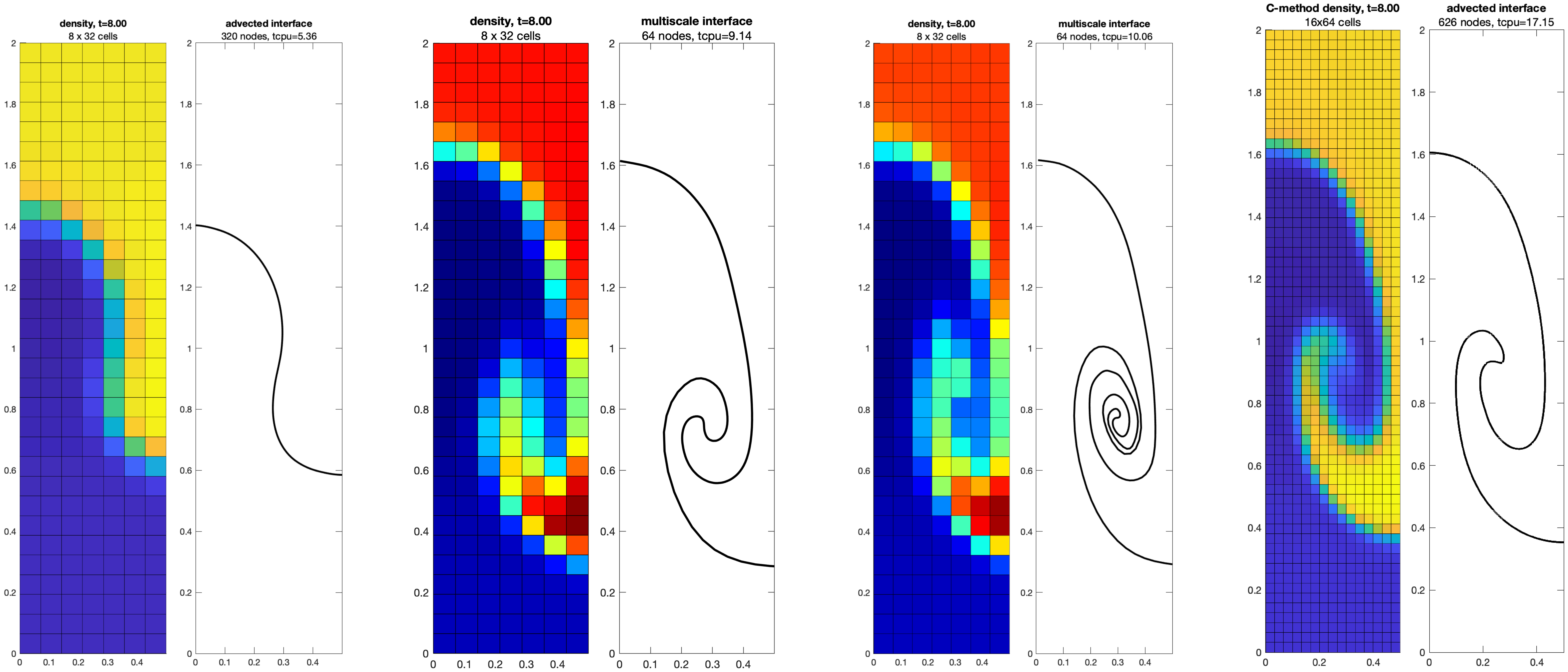
Full Multiscale System

- Bulk equations, for $(\rho, \rho\tilde{u}, \rho e)$:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho\tilde{u}) = -\nabla \cdot (\rho\bar{u}) \\ \partial_t(\rho\tilde{u}) + \nabla \cdot (\rho\tilde{u} \otimes \tilde{u}) + \nabla p = -\partial_t(\rho\bar{u}) - \nabla \cdot (\rho\bar{u} \otimes \bar{u} + \rho\bar{u} \otimes \tilde{u} + \rho\tilde{u} \otimes \bar{u}) \\ \partial_t(\rho e) + \nabla \cdot ((\rho e + p)\tilde{u}) = -\nabla \cdot ((\rho e + p)\bar{u}) \end{cases}$$

- Interfacial equations, for (z, μ):

$$\begin{cases} \partial_t z(s,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(z(s,t) - z(s',t))^{\perp}}{|z(s,t) - z(s',t)|^2} \mu(s',t) ds' + \tilde{u}(z(s,t),t), \\ \partial_t \mu = \left(\frac{\rho^+ - \rho^-}{\rho^+ + \rho^-} \circ z \right) \partial_s \left(|\partial_t z|^2 - \frac{\mu^2}{4|\partial_s z|^2} - 2g z_2 \right) \end{cases}$$

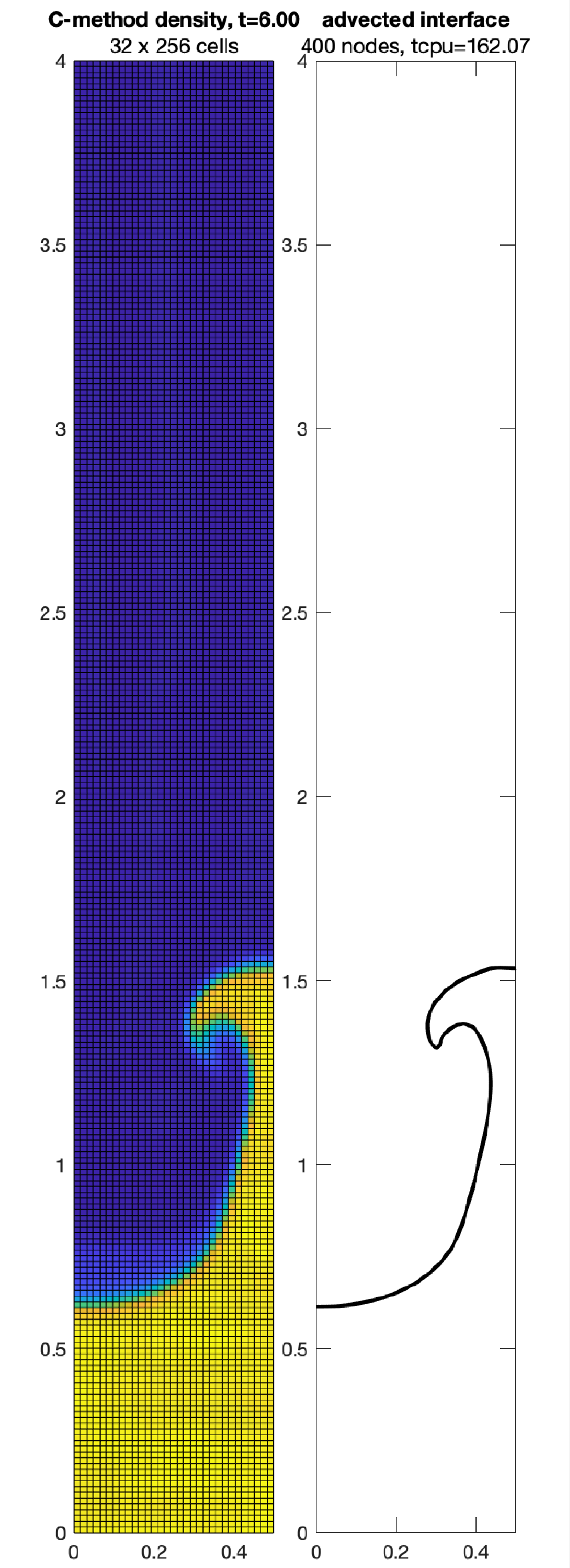


8x32 + front tracking

8x32 + constant-density multiscale

8x32 + variable-density multiscale

16x64 + front tracking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advected interface
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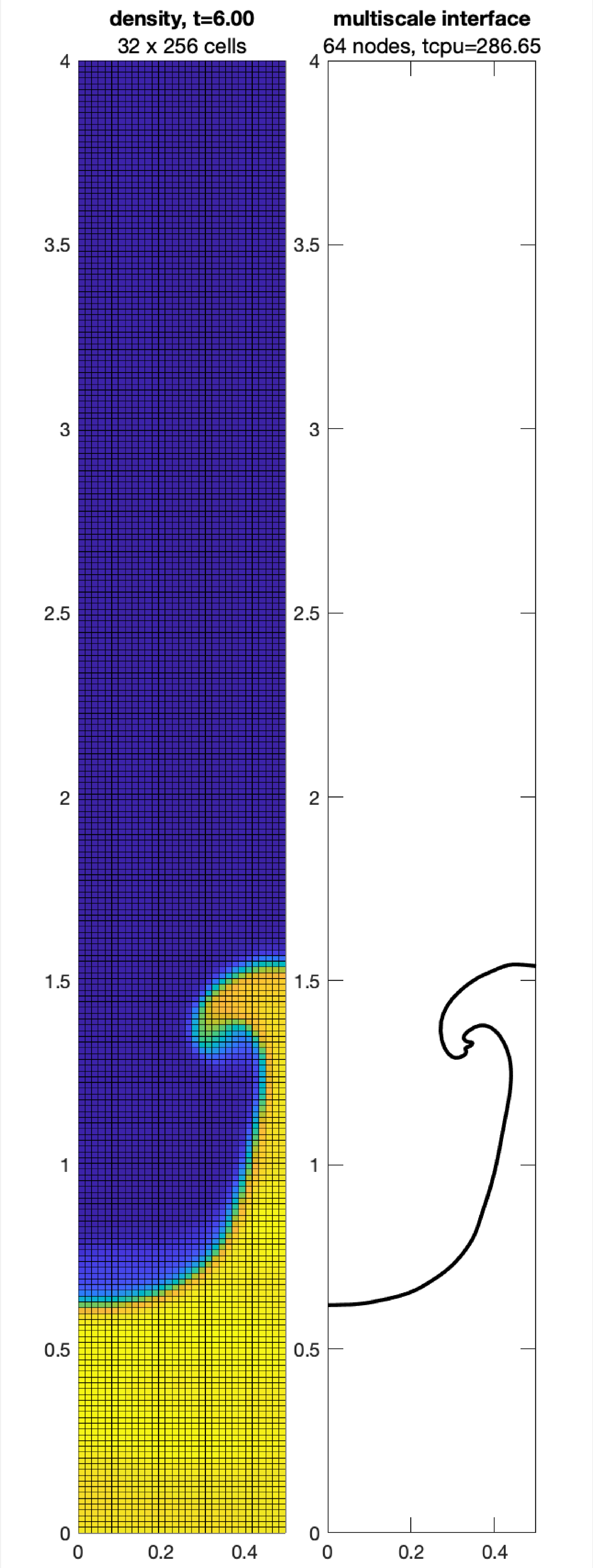
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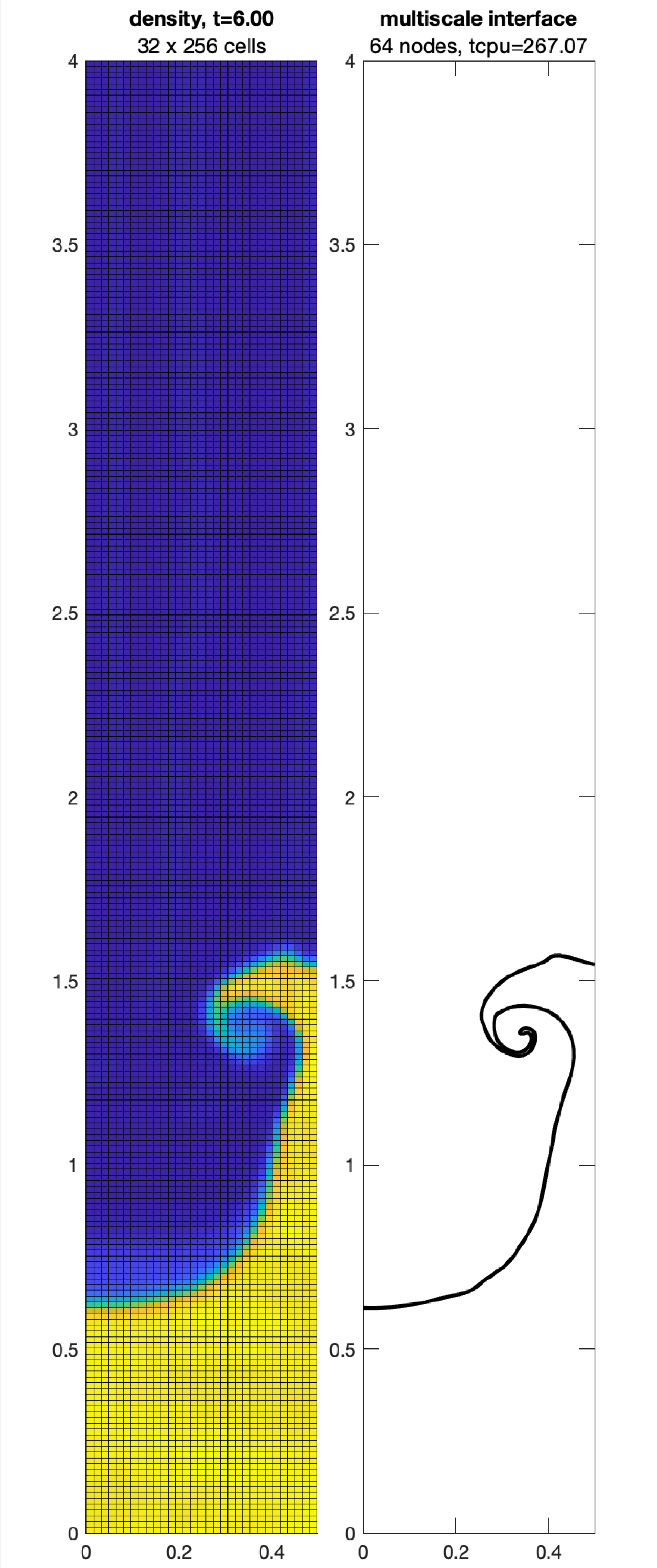
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multiscale interface
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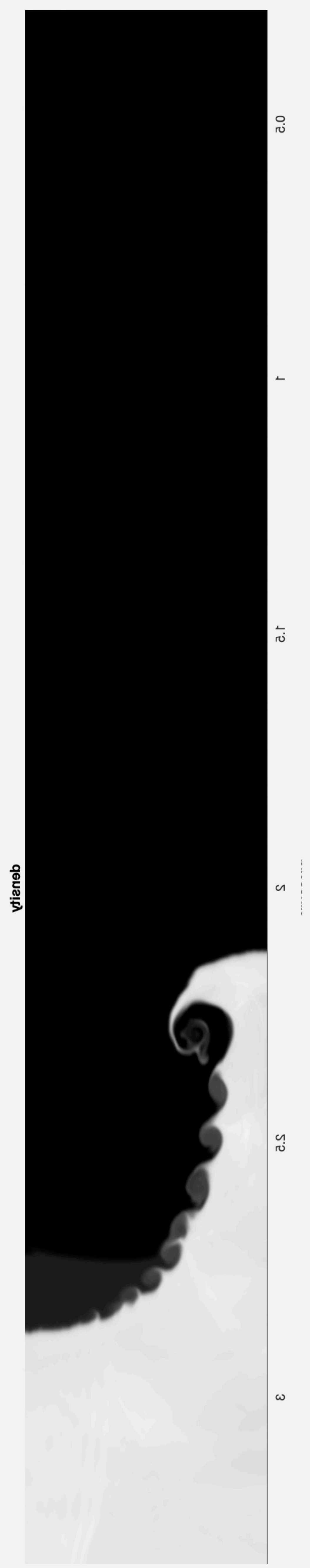

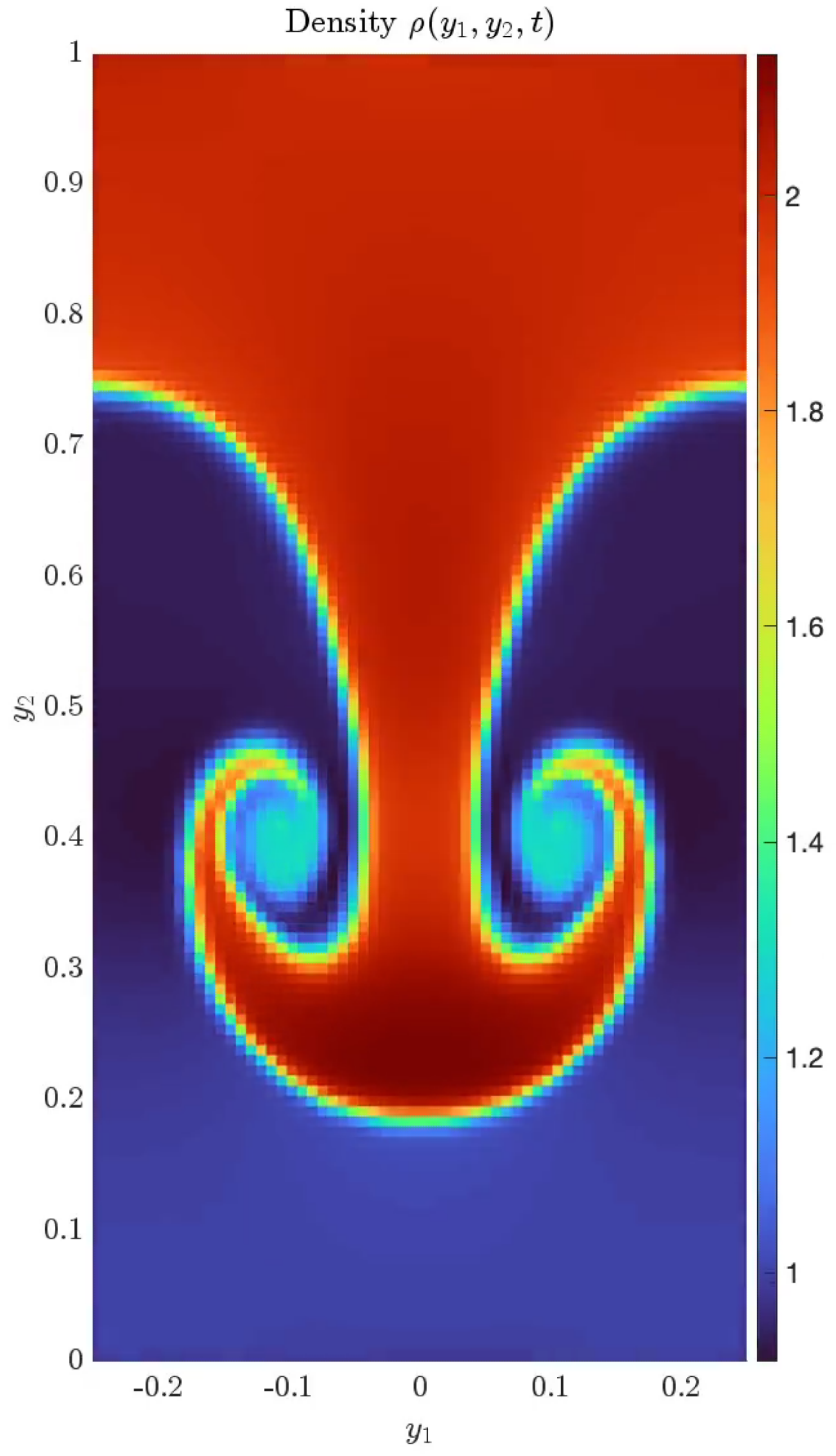

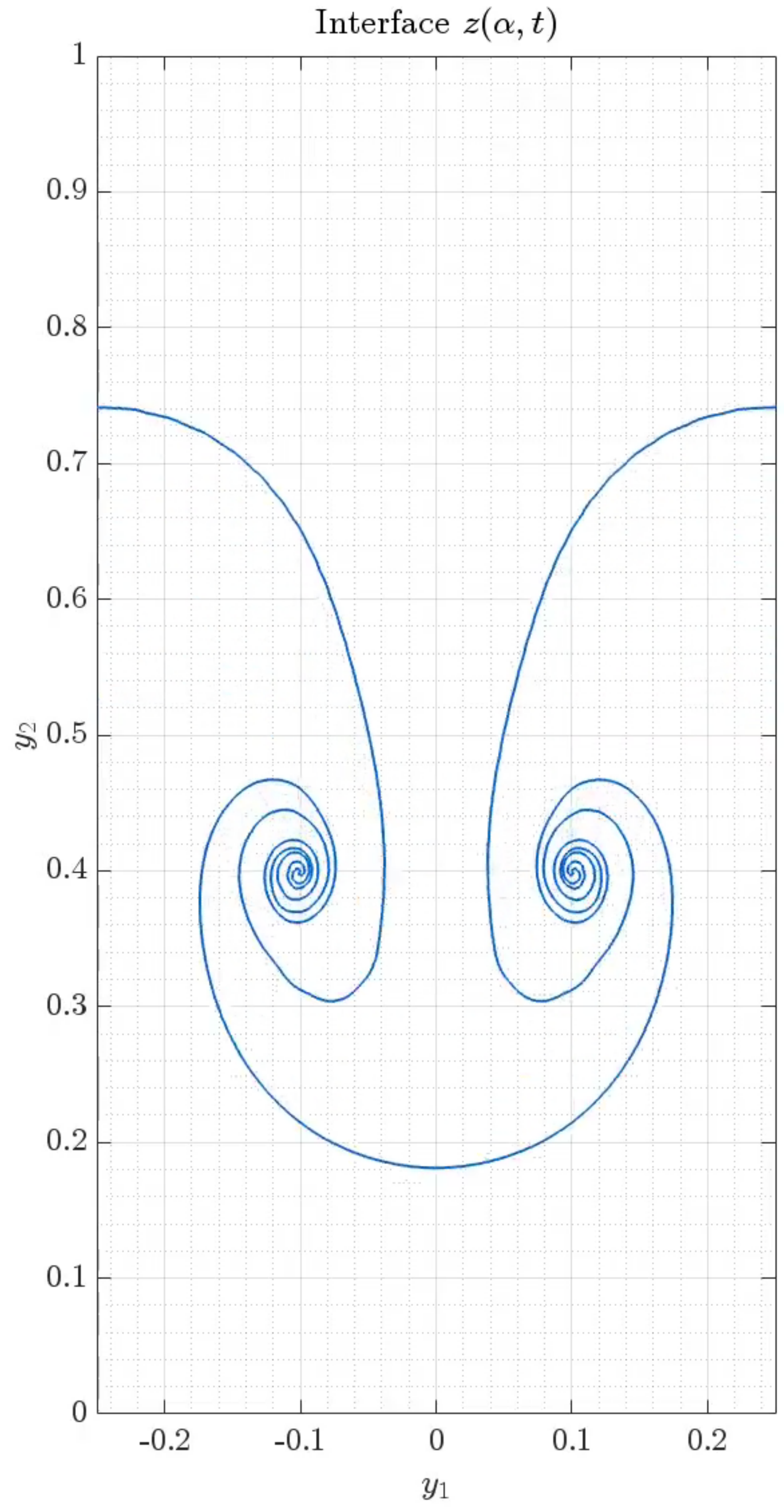

Multiscale RTI

Summary

- Developed hybrid gridded-interface model for contact discontinuities, generalizing preexisting z-model interface method.
- Found computational efficiency gains and good agreement with higher-resolution front tracking runs for Rayleigh-Taylor and Richtmyer-Meshkov problems.


Thank You!