HW 2 Solutions

4.5) 3g. Proof. We prove that \(x\) odd implies \(8 \mid (x^2 - 1)\).

If \(x\) is odd, there is a \(k \in \mathbb{Z}\) so that \(x = 2k + 1\). Then \(x^2 - 1 = 4k^2 + 4k + 1 - 1 = 4k(k + 1)\). Either \(k\) or \(k + 1\) is even, so \(4k(k + 1)\) is divisible by 8. \(\square\)

4c) Proof. Assume \(x \leq 0\). Then \(x^3 + x = x(x^2 + 1) \leq 0\), as \(x^2 + 1\) is positive. \(\square\)

5c) Proof. The distance \(d((2, 4), (0, 3)) = \sqrt{4 + 1} = \sqrt{5}\)

while \(d((2, 4), (3, 2)) = \sqrt{1 + 9} = \sqrt{10} > \sqrt{5}\). \(\square\)

6e) Proof. If \(a \geq 2\), \(a < b\) and \(ab < 3\), then \(b \geq 3\) (as \(b > a\)), and then \(ab \geq 6\). \(\square\)

7b) Proof. (\(\Leftarrow\)) Easy check.

(\(\Rightarrow\)) If \(b \mid (b+3)\), then \(b + 3 = lb\) for some \(l \geq 1\), but then \(3 = (l-1)b\) and \(b \mid 3\). Since \(3\) is prime, \(b = 1\) or \(b = 3\). But \(b = 1\) is impossible, as \((a + 1) \mid 1\) is impossible because \(a + 1 \geq 2\).

Therefore \(b = 3\); \(a + 1 \mid 3\) and \(a + 1 \geq 2\) then together imply \(a + 1 = 3\); \(a = 2\). \(\square\)

10. Omitted.

12(d) The proof is correct but unnecessarily convoluted. From the first two sentences, replace next "However" by "By the assumption" and shorten the last sentence to "Thus \(a\) divides \(b + c\)."
4(b) True. Take $y = -x$.

4(c) False. Counterexample: $x = 2$, $y = 1$.

4(g) False. Counterexample: $x = 1$.

5(a) Proof. ($\Rightarrow$) Contrapositive:

($\Leftarrow$) Also contrapositive. Assume $x$ is not a prime. Then $x = m \cdot n$ for some $m, n \in \mathbb{N}$.

At least one of $m, n$ must be less or equal $\sqrt{x}$. If both $m > \sqrt{x}$ and $n > \sqrt{x}$, then $m \cdot n > \sqrt{x} \cdot \sqrt{x} = x$, a contradiction with $m \cdot n = x$.

Assume that $m \leq \sqrt{x}$, without loss of generality. Then $m$ is a positive integer less or equal to $\sqrt{x}$ that divides $x$. \(\Box\)

6(c) $\frac{1}{r^2} < \frac{1}{100}$ is equivalent to $r > 10$.

So we can take $K = 10$. \(\Box\)

7(b) False. What we proved is the converse

$(x = 0) \lor (y = 0) \implies xy = 0$.

Finishing. The time is unbounded, but the